

F 1503



BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI

TOMUL XVII (XXI)

Fasc. 1—2

**SECȚIA I MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

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1971

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DEUX APPLICATIONS D'UN PROCÉDÉ DE CONSTRUCTION DE FONCTIONS

PAR

AL. CLIMESCU

1. Soient R l'ensemble des réels, N l'ensemble des entiers et $A_0, A_1, \dots, A_n$, la suite d'ensembles définie par

$$A_0 = N, \quad A_{n+1} = \frac{1}{2} A_n \quad (n = 0, 1, \dots).$$

La suite possède les propriétés suivantes :

(1) A_n est équidistante soit

$$A = \{nh\} \quad n \in N \quad h = \frac{1}{2^n};$$

(2) $A_0 \subset A_1 \subset \dots \subset A_n \subset \dots$;

(3) $A_n = \bigcup A$ est partout dense sur R .

Construisons la suite de fonctions $f_0(x), \dots, f_n(x) \dots$ de la manière suivante :

(4) choix arbitraire de valeurs $a_p = f_n(ph) \quad (n = 0, 1, \dots)$;

(5) interpolation linéaire entre ph et $(p+1)h$ pour p quelconque $\in N$;

La suite $f_n(x)$ détermine les fonctions $F(x)$ et $f(x)$ définies par

$$(6) \quad F(x) = \limsup f_n(x), \quad f(x) = \liminf f_n(x),$$

dont le domaine est R et le codomaine $R \cup \{\infty\} \cup \{-\infty\}$.

Le problème que nous nous posons est celui du choix des a_p de telle façon que les propriétés que peuvent avoir les $f_n(x)$ se reflètent clairement pour au moins l'une des deux fonctions (6). Ce problème, qu'il est difficile de

cerner de plus près peut être aussi formulé pour d'autres suites d'ensembles, par exemple pour

$$B_0 = N, \quad B_{n+1} = \frac{1}{n+1} B,$$

dont la réunion est l'ensemble des nombres rationnels.

On entrevoit tout de suite deux cas pour lesquels le problème est abordable; le premier cas se présente lorsque le choix des a_p implique $F(x) = f(x)$ et le second lorsque les a_p déterminent des points en lesquels les suites numériques correspondantes ne diffèrent que pour un nombre fini d'indices.

Nous voulons donner ici, pour la suite A_n un exemple pour chacun de ces cas.

2. Soit $a_p = (1+h)^p$ et soit $a = k/2^n$ un point quelconque de A .
On a

$$(7) \quad a = \frac{k}{2^n} = \frac{2k}{2^{n+1}} = \frac{4k}{2^{n+2}} = \dots$$

donc

$$f_n(a) = \left(1 + \frac{1}{2^n}\right)^k = \left(1 + \frac{a}{k}\right)^k, \quad f_{n+1}(a) = \left(1 + \frac{1}{2^{n+1}}\right)^{2k} = \left(1 + \frac{a}{2^k}\right)^{2k}, \dots$$

$$f_{n+p}(a) = \left(1 + \frac{a}{2^{pk}}\right)^{2^{pk}}, \dots$$

donc $\lim f_n(a) = e^a$ et il est clair qu'on a $\lim f_n(x) = e^x$; il est d'ailleurs facile de définir l'exponentielle à l'aide de ce processus et d'en déduire toutes ses propriétés essentielles.

3. Pour construire le seconde exemple, posons

$$(8) \quad \begin{cases} f_n(a) = 1 & \text{si } a = \frac{k}{2^n} \text{ et } k \text{ est pair;} \\ f_n(a) = -1 & \text{si } a = \frac{k}{2^n} \text{ et } k \text{ est impair.} \end{cases}$$

Si $\frac{k}{2^n} \leq x \leq \frac{(k+1)}{2^n}$ on aura :

$$(9) \quad \begin{cases} f_n(x) = -2^{n+1}x + 1 + 2k & \text{si } k \text{ est pair;} \\ f_n(x) = 2^{n+1}x - 1 - 2k & \text{si } k \text{ est impair.} \end{cases}$$

Les propriétés de la suite $f_n(a)$ sont alors les suivantes :

$$(10) \quad |f_n(x)| \leq 1, \quad (n = 0, 1, \dots),$$

$$(11) \quad f_n(-x) = f_n(x), \quad (n = 0, 1, \dots),$$

$$(12) \quad f_n(a) \text{ est stationnaire pour chaque } a \in A,$$

$$(13) \quad f_{n+1}\left(\frac{x}{2}\right) = f_n(x), \quad (n = 0, 1, \dots).$$

$$(14) \quad \text{Si } h = 2^{-n} \text{ et } m \geq n + 2, \text{ on a } f_m(x + h) = f_m(x).$$

$$\text{Pour (10): on a si } \frac{k}{2^n} \leq x \leq \frac{k+1}{2^n},$$

$$-1 \leq 2^{n+1}x - 1 - 2k \leq 1,$$

donc si k est quelconque $|f_n(x)| \leq 1$.

Pour (11)

$$\frac{k}{2^n} \leq x \leq \frac{k+1}{2^n} \text{ implique } -\frac{k+1}{2^n} \leq -x \leq -\frac{k}{2^n}.$$

Si k est pair on a pour les deux intervalles concernés $f_n(x) = -2^{n+1}x + 1 + 2k$ respectivement $f_n(x) = 2^{n+1}x - 1 - 2(-k-1) = 2^{n+1}x + 1 + 2k$ d'où si x est dans le premier intervalle

$$f_n(-x) = 2^{n+1}(-x) + 1 + 2k = f_n(x).$$

Résultat similaire pour k impair.

Pour (12): d'après (7) on a

$$f_{n+1}\left(\frac{k}{2^n}\right) = f_{n+2}\left(\frac{k}{2^n}\right) = \dots = 1.$$

Pour (13): si

$$\frac{k}{2^n} \leq x \leq \frac{k+1}{2^n} \text{ on a } \frac{k}{2^{n+1}} \leq \frac{x}{2} \leq \frac{k+1}{2^{n+1}};$$

si k est pair on a pour les deux intervalles

$$f(x) = -2^{n+1}x + 1 + 2k, \text{ respectivement } f(x) = -2^{n+1}x + 1 + 2k,$$

d'où pour un x du premier intervalle

$$f_{n+1}\left(\frac{x}{2}\right) = -2^{n+2}\frac{x}{2} + 1 + 2k = f_n(x).$$

Même résultat si k est impair.

Pour (14): prenons d'abord $n = 0$, donc il faut démontrer l'égalité

$$f_n(x + 1) = f_n(x) \text{ si } m \geq 2.$$

Soit $0 \leq x \leq 1$ et n un entier quelconque.

Premier cas : $0 \leq x \leq \frac{1}{4} \Rightarrow \frac{4n}{4} \leq x + n \leq \frac{4n+1}{4}$, donc pour les deux intervalles mis en évidence

$$f_2(x) = -8x + 1 \quad \text{et} \quad f_2(x) = -8x + 1 + 8n,$$

d'où si $0 \leq x \leq \frac{1}{4}$

$$f_2(x+n) = -8(x+n) + 1 + 8n = f_2(x).$$

Deuxième cas : $\frac{1}{4} \leq x \leq \frac{2}{4} \Rightarrow \frac{4n+1}{4} \leq x+n \leq \frac{4n+2}{4}$ et $f_2(x) = 8x - 3$, respectivement $f_2(x) = 8x - 1 - 2(4n+1) = 8x - 8n - 3$ et

$$f_2(x) = 8x - 3 \Rightarrow f_2(x+n) = 8(x+n) - 8n - 3 = f_2(x).$$

Démonstration identique pour $\frac{2}{4} \leq x \leq \frac{3}{4}$, respectivement $\frac{3}{4} \leq x \leq \frac{4}{4}$, et l'égalité $f_2(x+n) = f_2(x)$ (en particulier $f_2(x+1) = f_2(x)$ pour tout x en découle immédiatement).

En employant (13) sous la forme

$$f_3(x) = f_2(2x)$$

on a $f_3(x+1) = f_2(2x+2) = f_2(2x) = f_3(x)$ et par induction on démontre (15).

On a ensuite

$$f_3\left(x + \frac{1}{2}\right) = f_3\left(\frac{2x+1}{2}\right) = f_2(2x+1) = f_2(2x) = f_3(x)$$

et inductivement

$$f_m\left(x + \frac{1}{2}\right) = f_m(x) \quad \text{si} \quad m \geq 3.$$

De même on peut montrer inductivement que $f_m(x+h) = f_m(x)$ entraîne

$$f_{m+1}\left(x + \frac{h}{2}\right) = f_{m+1}(x)$$

pour $h = \frac{1}{2^n}$, $m = n+2$; par une nouvelle induction on achève la démonstration de (14).

La forme des résultats montre qu'ils se propagent à la limite donc on a la proposition suivante:

La fonction $F(x)$ (et de même $f(x)$) a les propriétés suivantes:

$$(15) \quad |F(x)| \leq 1,$$

$$(16) \quad F(-x) = F(x),$$

$$(17) \quad F_n(a) = 1 \text{ pour tout } a \in A,$$

$$(18) \quad F\left(\frac{x}{2}\right) = F(x),$$

$$(19) \quad F\left(x + \frac{1}{2^n}\right) = F(x), \quad (n = 0, 1, \dots).$$

Évidemment on peut prendre aussi $n \in N$ quelconque dans la formule (19) (18) montre que $F(x)$ est solution d'une équation fonctionnelle d'Abel (cf. [1, ch. VI] pour une étude approfondie de l'équation d'Abel).

4. Il reste à démontrer que $F(x)$ n'est pas constante. Or on a

$$\frac{1}{3} = \frac{1}{2^2} + \frac{1}{2^4} + \dots + \frac{1}{2^n} + \dots$$

et on en déduit

$$\frac{1}{4} < \frac{1}{3} < \frac{2}{4}, \quad \frac{2}{8} < \frac{1}{3} < \frac{3}{8},$$

donc pour les intervalles considérés:

$$f_2(x) = 8x - 3, \quad f_3(x) = -16x + 5$$

et

$$f_2\left(\frac{1}{3}\right) = -\frac{1}{3}, \quad f_3\left(\frac{1}{3}\right) = -\frac{16}{3} + 5 = -\frac{1}{3}.$$

Ensuite on a

$$f_n\left(\frac{1}{3}\right) = f_3\left(\frac{2}{3}\right) = f_3\left(-\frac{1}{3} + 1\right) = f_3\left(-\frac{1}{3}\right) = f_3\left(\frac{1}{3}\right) = -\frac{1}{3}$$

et plus généralement

$$f_{n+1}\left(\frac{1}{3}\right) = f_n\left(\frac{2}{3}\right) = f_n\left(-\frac{1}{3} + 1\right) = f_n\left(-\frac{1}{3}\right) = f_n\left(\frac{1}{3}\right), \quad (n \geq 2),$$

d'où par induction

$$f_n\left(\frac{1}{3}\right) = -\frac{1}{3}$$

$$\text{et } F\left(\frac{1}{3}\right) = f\left(\frac{1}{3}\right) = -\frac{1}{3}.$$

Corollaire. En tenant compte du fait que A est partout dense et qu'on a $F(x) = -\frac{1}{3}$ si $x \in \frac{1}{3} + A$, on voit que les points de discontinuité de $F(x)$ forment un ensemble partout dense; il est probable que $F(x)$ est partout discontinue.

Pour un x quelconque, qu'il suffit de supposer positif et < 1 , supposons qu'on ait

$$(20) \quad \frac{k}{2^n} \leq x \leq \frac{k+1}{2^n},$$

donc

$$(21) \quad \frac{2k}{2^{n+1}} \leq x \leq \frac{2k+1}{2^{n+1}}, \quad \text{ou} \quad \frac{2k+1}{2^{n+1}} \leq x \leq \frac{2k+2}{2^{n+1}}.$$

Un calcul simple montre alors qu'on a

$$f_{n+1}(x) = 2f_n(x) - 1 \quad \text{ou} \quad f_{n+1}(x) = -2f_n(x) - 1,$$

selon que les numérateurs à gauche dans (20) et (21) ont ou non même parité. Les différents cas qui peuvent se présenter sont d'ailleurs déterminés par le développement binaire de x .

Reçue le 26 novembre 1970

Institut Polytechnique, Jassy,
Chaire de Mathématiques I

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DOUĂ APLICAȚII ALE UNUI PROCEDEU DE CONSTRUCȚII DE FUNCȚII

(Rezumat)

Se studiază proprietățile unei funcții $F(x)$ definită prin (6), găsiindu-se că $F(x)$ posedă proprietățile (15)...(19). În particular rezultă că $F(x)$ este soluție a unei ecuații funcționale a lui Abel, punctele ei de discontinuitate formînd o mulțime peste tot densă.

ÉQUATIONS FONCTIONNELLES EN GÉOMÉTRIE
ET PROBABILITÉS ¹⁾

PAR

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1. Il est bien connu qu'un processus markovien continu simple à n états est caractérisé par une matrice de passage formée par des éléments non négatifs P_j^i

$$(1) \quad P(s, t) = \| P_j^i(s, t) \|, \quad (s \leq t),$$

où i, j varient de 1 à n . Cette matrice doit avoir les propriétés

$$(2) \quad \sum_{j=1}^n P_j^i(s, t) = 1, \quad P_j^i(s, s) = \delta_j^i,$$

où δ_j^i sont les symboles de Kronecker donc $\delta_j^i = 0$ si $i \neq j$ et $\delta_j^i = 1$ si $i = j$.

De plus les quantités P_j^i (les probabilités) doivent satisfaire aux équations fonctionnelles de Chapman—Kolmogoroff

$$(3) \quad P_j^i(s, t) = P_k^i(s, u) P_j^k(u, t), \quad (s \leq u \leq t).$$

On connaît d'après Fréchet une solution simple de ces équations fonctionnelles en prenant

$$(4) \quad P_j^i(s, t) = \mu_a^i(s) \lambda_j^a(t), \quad (a = 1, \dots, n),$$

où $\| \mu_a^i(s) \|$ est une matrice non singulière, donc le déterminant $\Delta(s) = | \mu_a^i(s) |$ est différent de zéro. Quant à $\| \lambda_j^a(t) \|$ elle est la matrice duale, donc nous avons les conditions

$$(5) \quad \mu_a^i(s) \lambda_j^a(s) = \delta_j^i.$$

¹⁾ Conférence faite à l'Université d'Alberta, Edmonton, Canada, le 27 avril 1970.

En interprétant s comme une coordonnée sur la droite réelle, on peut donc dire que les formules (4) nous donnent les probabilités $P_j^i(s, t)$ à l'aide de la matrice $\|\mu_a^i\|$ dans le point P de coordonnée s de cette droite et à l'aide de la matrice duale $\|\lambda_j^a\|$ dans le point de coordonnée t de la même droite, en supposant toujours que nous avons $s \leq t$.

2. Mon fils G. G. Vranceanu [1] et moi nous avons considérés cet problème dans un espace à n dimensions $V_n(s^1, \dots, s^n)$, donc s signifie l'ensemble $s^1, \dots, s^n$ et t l'ensemble $t^1, \dots, t^n$ et les conditions $s \leq t$ sont données par exemple par les conditions

$$(6) \quad s^1 \leq t^1, \quad s^2 \leq t^2, \dots, \quad s^n \leq t^n.$$

Cela a permis de donner des interprétations géométriques aux processus markoviens à n états dans des espaces à n dimensions. On a montré en particulier que l'on peut associer une connexion affine à l'espace $V_n(s^1, \dots, s^n)$ et le fait que les probabilités $P_j^i(s^1, \dots, s^n)$ sont positives se traduit par une propriété de connexion, donc par une propriété géométrique de l'espace V_n associé au processus markovien. Il est évident que si nous avons un certain système de probabilités $P_j^i(s^1, \dots, s^n; t^1, \dots, t^n)$ on peut obtenir un système de probabilités dans deux points s, t de la droite, en reliant les points $P(s^1, \dots, s^n)$ et $Q(t^1, \dots, t^n)$ de l'espace V_n par une courbe ou par une droite, par exemple en faisant

$$s^1 = \dots = s^n = s, \quad t^1 = \dots = t^n = t.$$

3. L'introduction de la notion de probabilité dans l'espace V_n permet d'obtenir des propriétés géométriques importantes de l'espace V_n lui-même et nous allons donner un seul exemple pour montrer qu'il existe des espaces de Riemann V_n dans lesquels on peut définir un système de probabilités valables dans tout l'espace. Pour cela soit la métrique du semi-plan de Poincaré

$$(7) \quad ds^2 = \frac{dx^2 + dy^2}{y^2},$$

donc la métrique de la géométrie non euclidienne de Lobachevsky-Bolyai.

Si $P(x, y)$ et $P'(x', y')$ sont dits points consécutifs, si nous avons à la place des formules (6), les formules

$$(8) \quad \frac{y}{1+e^x} \leq \frac{y'}{1+e^{x'}}, \quad \frac{ye^x}{1+e^x} \leq \frac{y'e^{x'}}{1+e^{x'}}$$

et l'on prend des probabilités sous la forme

$$(9) \quad \begin{cases} P_1^1 = \frac{1}{y'} \left[\frac{y' e^{x'}}{1 + e^{x'}} + \frac{y}{1 + e^x} \right], & P_2^1 = \frac{1}{y'} \left[\frac{y'}{1 + e^{x'}} - \frac{y}{1 + e^x} \right], \\ P_1^2 = \frac{1}{y'} \left[\frac{y' e^{x'}}{1 + e^{x'}} - \frac{y e^x}{1 + e^x} \right], & P_2^2 = \frac{1}{y'} \left[\frac{y'}{1 + e^{x'}} + \frac{y e^x}{1 + e^x} \right], \end{cases}$$

il en résulte que $P_1^1, P_2^1, P_1^2, P_2^2$ sont toutes positives, car y, y' sont positifs pour le semi-plan de Poincaré, et P_j^i sont définies dans tout ce semi-plan.

D'ailleurs en prenant en accord avec (8) comme nouvelles variables

$$(10) \quad u = \frac{y e^x}{1 + e^x}, \quad v = \frac{y}{1 + e^x},$$

ce qui nous donnent comme formules inverses

$$(11) \quad e^x = \frac{u}{v}, \quad y = u + v,$$

la métrique (7) s'écrit

$$(12) \quad ds^2 = \frac{v^2 (1 + u^2) du^2 + u^2 (1 + v^2) dv^2 + 2 du dv uv (uv - 1)}{u^2 v^2 (u + v)^2}$$

et cette métrique est régulière dans le quart du plan $u > 0, v > 0$. Dans ces nouvelles variables les probabilités s'écrivent sous la forme qu'on peut appeler canonique

$$(13) \quad \begin{cases} P_1^1 = \frac{u' + v}{u' + v'}, & P_2^1 = \frac{v' - v}{u' + v'}, \\ P_1^2 = \frac{u' - u}{u' + v'}, & P_2^2 = \frac{v' + u}{u' + v'}. \end{cases}$$

La notion d'ordre étant dans ce cas

$$(14) \quad u \leq u', \quad v \leq v',$$

les quantités u, v, u', v' étant positives.

On peut montrer que la propriété s'étend aux espaces V à courbure constante négative

$$(15) \quad ds^2 = \frac{(dx^1)^2 + \dots + (dx^n)^2}{(y^n)^2},$$

qui contient la métrique (7) comme cas particulier pour $n=2$, mais la propriété n'a pas lieu pour les espaces à courbure constante positive, comme c'est la sphère.

4. Nous avons montré jusqu'ici le profit que peut tirer la géométrie de la notion de probabilité dans un espace V_n à un nombre quelconque de dimensions. Nous allons maintenant montrer comment certains problèmes géométriques peuvent conduire à des processus Markov, plus précisément à une classe particulière, les processus Poisson. On obtient de telles processus en supposant que les fonctions $P_j^i(s, t)$ sont additives, donc dépendent seulement de la somme $s+t$. Dans l'espace à n dimensions les P_j^i doivent être évidemment fonctions des variables $s^1+t^1, s^2+t^2, \dots, s^n+t^n$. En effet, dans l'étude des groupes commutatifs linéaires, donc des groupes commutatifs de l'espace affine, on est conduit à des équations fonctionnelles d'un type différent de celui du calcul de probabilités.

En effet on peut montrer qu'un sous-groupe commutatif Q_r du groupe linéaire Q_n ($r \leq n$) peut être réduit à une forme canonique en certaines variables se transforment d'après les formules

$$(16) \quad \begin{cases} x'^1 = x^1 + a^1, & x'^2 = x^2 + a^1 x^1 + a^2, \\ x'^s = x^s + a^1 x^{s-1} + \dots + a^{s-1} x^1 + a^s, & (s = 3, \dots, m), \end{cases}$$

où $a^1, \dots, a^m$ sont les paramètres du groupe et certaines variables disons $z^0, \dots, z^p$ se transforment d'après les formules

$$(17) \quad \begin{cases} z'^0 = b^0 z^0, & z'^1 = b^0 z^1 + b^1 z^0, \\ z'^k = b^0 z^k + b^1 z^{k-1} + \dots + b^k z^0, & (k = 2, \dots, p), \end{cases}$$

où $b^0, \dots, b^p$ sont les paramètres du groupe.

S'il y a seulement des transformations du type (16) le sous-groupe Q_r actionne dans tout l'espace euclidien affine $E_n(x^1, \dots, x^n)$, tandis que s'il y a des transformations (17) le sous-groupe Q_r a des points fixes; ce sont les points des variétés linéaires $z^0=0, z^0=z'=0, \dots, z^0=z^1=\dots=z^p=0$.

Cela dit, étant donné le fait que (16) où (17) sont des groupes commutatifs, trouver une transformation de variables de façon à transformer ces groupes dans le groupe des translations. Autrement dit il s'agit de trouver dans le cas (16) une transformation passant des coordonnées $x^1, \dots, x^m$ en des coordonnées $y^1, \dots, y^m$ de façon que dans ces nouvelles variables le groupe (16) soit donné par les formules

$$(18) \quad y'^s = y^s + \alpha^s, \quad (s = 1, \dots, m).$$

Or si l'on cherche cette transformation de coordonnées sous la forme

$$(19) \quad \begin{cases} x^s = R^s(y^1, \dots, y^s), \\ \alpha^s = R^s(\alpha^1, \dots, \alpha^s), \end{cases} \quad (s = 1, \dots, m),$$

on trouve facilement que les fonctions R^s doivent satisfaire les équations fonctionnelles

$$(20) \quad R^s(y^1 + \alpha^1, \dots, y^s + \alpha^s) = R^s(y^1, \dots, y^s) + R^s(\alpha^1, \dots, \alpha^s) + \\ + \sum_{k=1}^{s-1} R^k(\alpha) R^{s-k}(y), \quad (s = 1, \dots, m).$$

De même si dans le cas (17) on cherche le changement de coordonnées sous la forme

$$(21) \quad \begin{cases} z^s = f^s(u^0, \dots, u^s), \\ b^s = f^s(\beta^0, \dots, \beta^s), \end{cases} \quad (s=0, \dots, p),$$

on trouve les équations fonctionnelles

$$(22) \quad f^s(u^0 + \beta^0, \dots, u^s + \beta^s) = \sum_{k=0}^s f^k(\beta^0, \dots, \beta^k) f^{s-k}(u^0, \dots, u^k).$$

Cela dit dans le cas des équations fonctionnelles (20) on peut montrer que les R^s sont des polynômes et que l'on peut avoir une solution particulière en prenant

$$(23) \quad R^s = \sum \frac{(y^1)^{p_1}}{p_1!} \dots \frac{(y^s)^{p_s}}{p_s!},$$

où $p_1, \dots, p_s$ sont des nombres entiers non négatifs et où la somme s'étend à toutes les solutions de l'équation diophantienne

$$(24) \quad p_1 + 2p_2 + \dots + sp_s = s.$$

Dans le cas des équations fonctionnelles (22) la solution est donnée aussi par des polynômes mais multipliées par le facteur $e^{-\lambda u^0}$, étant donné que la première équation (22) s'écrit

$$(25) \quad f^0(u^0 + \beta^0) = f^0(\beta^0) f^0(u^0),$$

qui a comme solution

$$(26) \quad f^0 = e^{-\lambda u^0}.$$

On a donc aussi dans le cas (22) une solution particulière en prenant

$$(27) \quad z^0 = e^{-\lambda u^0}, \quad z^s = e^{-\lambda u^0} R^s, \quad (s = 1, \dots, p),$$

où R^s sont les polynômes (23) dans les variables $u^1, \dots, u^s$.

Maintenant si l'on se restreint à une seule variable en passant par exemple

$$(28) \quad u^0 = t, \quad u^s = c_s t, \quad t > 0,$$

où c_s sont des constantes positives, les formules (27) conduisent aux processus de Poisson étudiés par Janossy, Renyi et Aczel [2].

Reçue le 11 novembre 1970

Institut de Mathématiques de l'Académie
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ECUAȚII FUNCȚIONALE ÎN GEOMETRIE ȘI PROBABILITĂȚI

(Rezumat)

Utilizând rezultatele studiilor privitoare la procese markoviene și la ecuațiile funcționale ale lui Chapman-Kolmogorov, se prezintă o interpretare geometrică a proceselor markoviene avînd n stări în spații cu n dimensiuni și se arată, printre altele, cum unele probleme geometrice pot conduce la procese Markov.

ON A COLOURING PROBLEM OF STRONG
CONNECTED GRAPHS

BY

CORINA REISCHER and DAN A. SIMOVICI

1. Introduction

In this paper there is given a necessary and sufficient condition for the existence of a special colouring property of finite graphs. Namely, we wish to find out a partition $\{G_1, \dots, G_p\}$ of the vertices of a finite graph which enjoys the property that the image of the set G_i is G_{i+1} and the image of G_p is G_1 . We give also some results concerning the incidence matrix of the graph, and an algorithm, based on the Boolean matricial techniques, which performs the finding of the sets $G_1, \dots, G_p$. The interest of this problem consists in its application in automata theory.

2. A Colouring Property of Strong Connected Graphs

The main result of this part is included in the following

Theorem 1. *Let $\mathcal{G} = (G, \Gamma)$ a finite, strong connected graph for which there exists a natural number p so that the length of each circuit is a multiple of p . Then we can colour the graph $\mathcal{G}$ with p colours so that if G_i is the set of the vertices coloured with the colour i we have $\Gamma G_i = G_{i+1}$, $\Gamma G_p = G_1$.*

Proof. Let $g \in G$ a vertex of the graph $\mathcal{G}$.

Let us consider the sets

$$(1) \quad G_k = \bigcup_{\substack{i \in N \\ i \equiv k \pmod{p}}} \Gamma^i g = \bigcup_{m \in N} \Gamma^{mp+k} g, \quad k = \overline{1, p}.$$

The sets $\{G_k\}$, $k = \overline{1, p}$ are a partition of the set G of the vertices.

Firstly, $G_k \neq \emptyset$, $k = \overline{1, p}$ because the graph $\mathcal{G}$ is strong connected. We shall prove that $G_k \cap G_h = \emptyset$ for $k \neq h$. Indeed, if $g' \in G_k \cap G_h$ then, because $g' \in G_k$ there exists a path of length $mp + k$ from g to g' and due to the fact that $g' \in G_h$, there exists also a path from g to g' whose length is $m_1p + h$. Since $\mathcal{G}$ is a strong connected graph there exists a path from g' to g whose length is v . Then we obtain in the graph $\mathcal{G}$ two circuits having the length $v + mp + k$ and $v + m_1p + h$.

Because

$$v + mp + k \equiv 0 \pmod{p}, \quad v + m_1p + h \equiv 0 \pmod{p},$$

it is possible to write

$$v + k \equiv 0 \pmod{p}, \quad v + h \equiv 0 \pmod{p}$$

or

$$k - h \equiv 0 \pmod{p}.$$

Since $1 \leq k, h \leq p$, we obtain $k = h$, which contradicts the initial assumption.

Due to the strong connexity of the graph, for each vertex g there exists an index i for which $g \in G_i$. Finally, we must prove that $G_k = G_{k+1}$, $k = \overline{1, p}$, ($G_{p+1} = G_1$).

Indeed:

$$\Gamma G_k = \Gamma \left(\bigcup_{\substack{i \in N \\ i \equiv k \pmod{p}}} \Gamma^i g \right) = \bigcup_{\substack{i \in N \\ i \equiv k \pmod{p}}} \Gamma^{i+1} g = \bigcup_{\substack{j \in N \\ j \equiv k+1 \pmod{p}}} \Gamma^j g = G_{k+1}.$$

Remark 1. The sets G_k are internally stable.

Indeed

$$G_k \cap \Gamma G_k = G_k \cap G_{k+1} = \emptyset.$$

Remark 2. Let $\mu(g', g'')$ be the length of a path from g' to g'' . If the conditions included in Theorem 1 are fulfilled, we have

$$\mu(g', g'') + \mu(g'', g') \equiv 0 \pmod{p},$$

or

$$(2) \quad \mu(g', g'') \equiv -\mu(g'', g') \pmod{p}.$$

If $g_1, \dots, g_n$ is a sequence of vertices it is possible to write:

$$(3) \quad \mu(g_1, g_2) + \mu(g_2, g_3) + \dots + \mu(g_n, g_1) \equiv 0 \pmod{p}.$$

Remark 3. Let G_k be a set of the partition, obtained by the formula (1). Then g' belongs to G_k if:

$$(4) \quad \mu(g, g') \equiv k \pmod{p}.$$

Theorem 2. *The partition of the set of the vertices G is independent by the choice of the initial vertex g .*

In order to prove this, let g_1 and g_2 be two vertices and $\{G_k^1\}, \{G_k^2\}$, $k = \overline{1, p}$, the partitions related respectively by g_1 and g_2 .

Then:

$$G_u^1 = G_v^2,$$

where $u \equiv v + m \pmod{p}$; here $m = \mu(g_1, g_2)$.

Let $g \in G_u^1$. Taking into account the congruence (4) we have $\mu(g_1, g) \equiv u \pmod{p}$.

By using (3) for the sequence of vertices g_2, g, g_1 ,

$$\mu(g_2, g) + \mu(g, g_1) + \mu(g_1, g_2) \equiv 0 \pmod{p},$$

or, taking into account (2),

$$\begin{aligned} \mu(g_2, g) &\equiv -\mu(g, g_1) - \mu(g_1, g_2) \equiv \\ &\equiv \mu(g_1, g) - \mu(g_1, g_2) \equiv u - m \pmod{p}. \end{aligned}$$

Hence, on account of the validity of Remark 3, $g \in G_v^2$, where $v \equiv u - m \pmod{p}$ or $u \equiv v + m \pmod{p}$. It follows that $G_u^1 \subseteq G_v^2$. The converse inclusion can be proved in the same manner.

Remark 4. Theorem 1 can be extended immediately to an infinite, progressively-finite graph.

Remark 5. Assume that the conditions of the Theorem 1 are fulfilled for the number p . Then the same conditions are fulfilled for each divisor $d > 1$ of p and the graph $\mathcal{G}$ is d -colourable.

3. An Algorithm which Performs the Search of the Sets of the Partition

Let $\mathcal{G}$ a finite, strong-connected graph for which there exists a number p so that the length of each circuit is a multiple of p . This property can be checked using the modified algorithm of the latin multiplication [1], [3].

If H is a subset of the finite set $G = \{g_1, \dots, g_n\}$ we shall denote by χ_H its characteristic vector $\chi_H \in \mathcal{B}_2^{(n,1)}$ where

$$(5) \quad (\chi_H)_i = \begin{cases} 1, & \text{if } g_i \in H \\ 0, & \text{if } g_i \notin H. \end{cases}$$

Here $\mathcal{B}_2^{(p,q)}$ is the Boole algebra of the matrices with p rows and q columns whose entries belong to the two-element Boolean algebra $\mathcal{B}_2 = \{0, 1, \cup, \cap, -\}$.

Let $\mathbf{E}$ be the unit matrix, $\mathbf{E} \in \mathcal{B}_2^{(n,n)}$ and $\mathbf{E}_i, i = \overline{1, n}$ its columns. Then

$$\chi_{\{g_i\}} = \mathbf{E}_i, \quad i = \overline{1, n}.$$

Let $\mathbf{K}$ be the incidence matrix, of the graph $\mathcal{G}$. $\mathbf{K}$ is defined as follows: $\mathbf{K} = (k_{ij}) \in \mathcal{B}_2^{(n,n)}$

$$(6) \quad k_{ij} = \begin{cases} 1, & \text{if } g_j \in \Gamma g_i, \\ 0, & \text{if } g_j \notin \Gamma g_i. \end{cases}$$

It is easy to see that

$$(7) \quad \chi_{\Gamma H} = \mathbf{K} \chi_H.$$

Generally

$$(8) \quad \chi_{\Gamma^s H} = \mathbf{K}^s \chi_H.$$

The power of the Boolean matrix $\mathbf{K}$ is taken in the common way, namely:

$$k_{ij}^{(s)} = \bigcup_{p=1}^n k_{ip}^{(s-1)} k_{pj},$$

where $\mathbf{K}^s = (k_{ij}^{(s)})$.

Because the graph is strong connected, for every $g \in G$ there exists a circuit C_g whose length is a multiple of p which connects g to g . Let m_g be the minimal length of a circuit, going from g to g . If m is the least common multiple of the numbers $\{m_g, g \in G\}$ we have

$$(9) \quad \forall g \in G, \quad g \in \Gamma^m g.$$

Hence $\mathbf{E} \leq \mathbf{K}^m$, and using L u n c's theorem [2], there exists a natural number $e \leq n - 1$ for which

$$\mathbf{E} \leq \mathbf{K}^m \leq \mathbf{K}^{2m} \leq \dots \leq \mathbf{K}^{me} = \mathbf{K}^{m(e+1)} = \dots$$

But

$$\begin{aligned} \chi_{G_k} &= \bigcup_{\substack{i \in N \\ i \equiv k \pmod{p}}} \Gamma^i \chi_g = \bigcup_{\Gamma \in N} \mathbf{K}^{\Gamma p + k} \chi_g = \mathbf{K}^k \bigcup_{\Gamma \in N} \mathbf{K}^{\Gamma p} \chi_g = \\ &= \mathbf{K}^k \mathbf{K}^{me} \chi_g = \mathbf{K}^{me+k} \chi_g, \quad k = \overline{1, p}. \end{aligned}$$

Particularly,

$$\chi_{G_p} = \mathbf{K}^{me} \chi_g.$$

Hence, when $g = g_d$ the characteristic vector of the set G_k will be the d -th column of the matrix $\mathbf{K}^{me+k}$, $k = \overline{1, p}$.

4. Some Properties of the Incidence Matrix of the Graph $\mathfrak{G}$

Remark 6. It is clear that, for every k the matrix K^{me+k} can not have more than p distinct columns, where p is the greatest common divisor of the numbers $\{m_g, g \in G\}$.

Remark 7. The set of the i -th columns of the matrices $K^{m_1e}, K^{m_1e+1}, \dots, K^{m_1e+p-1}$ coincides, until the order with the set of the j -th columns of the matrices $K^{m_2e}, K^{m_2e+1}, \dots, K^{m_2e+p-1}$, where m_1 and m_2 are dividible by m .

Indeed, it is obvious that (9) holds for every m_1 dividible by m . Taking into account Theorem 2 and effecting the partition beginning with the i -th or the j -th vertex we obtain the previous result.

Remark 8. If we assume that the condition of Theorem 1 are satisfied, the set of the j -th columns from the matrices $K^{pe}, K^{pe+1}, \dots, K^{pe+e-1}$ can be obtained from the set of the first columns through a permutation.

The results obtained here can be extended for connected graphs and to general graphs.

Received November 5, 1970

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ASUPRA UNEI PROBLEME DE COLORARE A GRAFURILOR TARI CONEXE

(Rezumat)

Articolul cuprinde o caracterizare a grafurilor tari conexe care admit o proprietate specială de colorare. Astfel, fie graful $\mathfrak{G} = (G, \Gamma)$ tare conex și $\{G_1, \dots, G_p\}$ o partiție a vîrfurilor grafului. Teorema 1 oferă condiții necesare și suficiente pentru existența relațiilor $\Gamma G_i = G_{i+1}$, $\Gamma G_p = G_1$. Se demonstrează apoi unicitatea acestei partiții și se dă un algoritm bazat pe tehnici matriciale booleene care permite identificarea mulțimilor G_i . Problema este importantă pentru determinarea analogului periodic al unui automat finit dat.

ASUPRA UNUI MOD DE APROXIMARE A RĂDĂCINII UNEI ECUAȚII

DE

M. ȘT. BOTEZ

1. Pentru calculul cu aproximație a unei rădăcini a unei ecuații, se cunoaște un număr important de metode dintre care [1] cităm: metoda Newton și metoda aproximărilor succesive, care implică cunoașterea unei valori apropiate de aceea a rădăcinii căutate.

Alte metode ca aceea a lui Schröder consideră că $f(x)$ este analitică și desfășurabilă în serie într-un oarecare cerc de convergență în care se caută zerourile simple. König prezintă o metodă asemănătoare aceleia lui Schröder considerind însă că în cercul de convergență nu există decât un singur zero simplu, iar Rutishauser [2] îmbină metodele König și Bernoulli. O metodă deosebită este aceea a lui Aitken, numită încă și procesul δ^2 care corespunde sistemului:

$$x_{i+1} = \varphi(x_i), \quad x_{i+2} = \varphi(x_{i+1}), \quad x_{i+3} = \frac{x_i x_{i+2} - x_{i+1}^2}{x_i - 2x_{i+1} + x_{i+2}},$$

unde pentru ca să apară partea principală a ultimei relații, se introduc operatorii de diferență Δ : $\Delta x_i = x_{i+1} - x_i$, $\Delta^2 x_i = \Delta x_i = x_{i+2} - 2x_{i+1} + x_i$.

La cele citate se pot adăuga și alte metode, urmărind toate o îmbunătățire a calculului în condiția dificilă a necunoașterii unei metode exacte.

Corectarea posibilităților cunoscute pentru a găsi o valoare inițială astfel ca abaterea cât și numărul iterațiilor să fie cât mai mic, este din punct de vedere practic necesară.

Aceasta este ceea ce ne propunem în următoarele rânduri, fără însă a considera problema ca deplin rezolvată.

2. Fie ecuația:

$$(1) \quad f(x) = 0,$$

unde $f(x)$ este o funcție reală sau complexă, iar x variabilă reală.

Facem ipoteza că ecuația (1) are în intervalul (a, b) o singură rădăcină reală λ , cu $a < b$.

Se spune că o funcție f satisface condiția Lipschitz [3] de ordin α în x_0 , dacă există un număr pozitiv M și un interval $I =]x_0 - \alpha, x_0 + \alpha[$ cu $\alpha > 0$, astfel încît:

$$(2) \quad \forall x \in I \Rightarrow |f(x_1) - f(x_0)| < M |x_1 - x_0|^\alpha.$$

Se demonstrează că funcția f care verifică relația (2) este continuă și dacă $\alpha = 1$, nu are diferențială nulă.

Punctele x_1 și x_0 sînt conținute în intervalul I . Prin aceasta se operează o contractare a intervalului, ceea ce scurtează plaja iterării.

Dacă se consideră diferențele între rădăcina λ și limitele a și b ale intervalului:

$$(3) \quad \lambda - a = \mu, \quad b - \lambda = \nu,$$

atunci inegalitatea Lipschitz se poate scrie:

$$(4) \quad |f(\mu + a) - f(a)| < \mu M.$$

Însă $f(\mu + a) = f(\lambda) = 0$, deoarece λ este rădăcină. În acest caz relația (4) devine:

$$(5) \quad |f(a)| < \mu M,$$

sau $0 < |f(a)|/M < \mu$.

Considerînd cea de a doua egalitate dată de (3), se obține:

$$(5') \quad 0 < \frac{|f(b)|}{M} < \nu.$$

Cu ajutorul relațiilor (3), (5) și (5') se pot găsi două numere: a_1 cuprins între a și λ , și b_1 cuprins între λ și b , prin relațiile:

$$(6) \quad a_1 = a + \frac{|f(a)|}{M} \quad \text{și} \quad b_1 = b - \frac{|f(b)|}{M}.$$

Dacă se notează acum cu M_1 constanta Lipschitz pentru intervalul în prima sa contractare (a_1, b_1) , se pot exprima alte două numere a_2 și b_2 astfel încît relațiile (6) devin:

$$a_2 = a_1 + \frac{|f(a_1)|}{M_1} \quad \text{și} \quad b_2 = b_1 - \frac{|f(b_1)|}{M_1},$$

de unde rezultă recurențele:

$$(7) \quad a_i = a_{i-1} + \frac{|f(a_{i-1})|}{M_{i-1}} \quad \text{și} \quad b_i = b_{i-1} - \frac{|f(b_{i-1})|}{M_{i-1}}.$$

Se observă imediat că șirul (a_i) este crescător, iar (b_i) descrescător, avînd amîndouă limită.

Funcția $f(x)$ fiind continuă, anulându-se pentru o singură valoare λ , rezultă că cele două șiruri (a_i) și (b_i) au limită comună, care este rădăcina λ a ecuației $f(x) = 0$.

Se știe că pentru constantele M_i relative la intervalele (a_i, b_i) se pot lua numerele pozitive superioare tuturor valorilor ce le ia $|f'(x)|$ cuprinse între a_{i-1} și b_{i-1} .

Dacă $|f'(x)|$ este funcție reală crescătoare atunci M_{i-1} se poate înlocui prin $|f'(b_{i-1})|$ și relațiile (7) devin:

$$(7') \quad a_i = a_{i-1} + \frac{|f(a_{i-1})|}{|f'(b_{i-1})|}, \quad b_i = b_{i-1} - \frac{|f(b_{i-1})|}{|f'(b_{i-1})|},$$

iar dacă $|f'(x)|$ este descrescătoare în intervalul considerat, se ia pentru constanta iterată Lipschitz de ordinul $i-1$: $M_{i-1} = |f'(a_{i-1})|$, iar iterațiile (7) se exprimă prin

$$(7'') \quad a_i = a_{i-1} + \frac{|f(a_{i-1})|}{|f'(a_{i-1})|} \quad \text{și} \quad b_i = b_{i-1} - \frac{|f(b_{i-1})|}{|f'(a_{i-1})|},$$

formule care reamintesc iterațiile Newton, ca formă.

Primită la 14 iulie 1970

Institutul politehnic „Gh. Gheorghiu-Dej”,
București

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SUR UN PROCÉDÉ D'APPROXIMATION DES RACINES D'UNE ÉQUATION

(Résumé)

On présent un procédé qui permet d'approximer la racine d'une équation, par réduction de l'intervale qui contient la racine à l'aide de deux séries convergentes vers cette racine.

PROBLEMS CONCERNING POLYVIBRATING EQUATIONS

I. SOLUTIONS IN THE FORM OF AN F -FUNCTION

BY

D. MANGERON and M. N. ÖGUZTÖRELI

To JOHN R. McGREGOR on his 45th birthday

1. Since the work [1], [2] of the first of the authors, a large number of contributions have been devoted to the study of polyvibrating and generalized polyvibrating equations [3] ... [5]. After an elegant representation formula due to M. Picone [6], these equations proved useful in mathematical physics [7]. In recent works on spline functions and the draftsman's equation, polyvibrating equations play an important role [8], [9]. A variety of boundary value problems, optimization problems and integro-differential equations involving polyvibrating operators have been considered by the present authors and by L. E. Krivosheina [10], [11].

In one of our very recent papers presented by G. Birkhoff for publication in the *Proceedings of the National Academy of Sciences of the U.S.A.* and considered by the experts whom he has consulted to be valuable and new we investigate solutions to the Darboux problem for a polyvibrating equation in the form of an F -function (C. Truesdell's approach).

In this paper, following the results of our paper mentioned above, we establish the analytic solution of the simple polyvibrating equation

$$(1.1) \quad \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} - \lambda u(x, y) = f(x, y)$$

satisfying the conditions

$$(1.2) \quad \begin{aligned} u(x, y_0) &= g(x), & u(x_0, y) &= h(y), \\ \frac{\partial^2 u(x, y)}{\partial x \partial y} \Big|_{y=y_0} &= p(x), & \frac{\partial^2 u(x, y)}{\partial x \partial y} \Big|_{x=x_0} &= q(y), \end{aligned}$$

where x and y are real variables, λ is a real parameter such that $|\lambda| < 1$, and $f(x, y)$, $g(x)$, $h(y)$, $p(x)$, and $q(y)$ are certain given functions represented by the power series

$$(1.3) \quad \left\{ \begin{array}{l} f(x, y) = \sum_{m, n=0}^{\infty} f_{m, n} \frac{(x - x_0)^m}{m!} \frac{(y - y_0)^n}{n!}, \\ g(x) = \sum_{m=0}^{\infty} g_m \frac{(x - x_0)^m}{m!}, \quad h(y) = \sum_{n=0}^{\infty} h_n \frac{(y - y_0)^n}{n!}, \\ p(x) = \sum_{m=0}^{\infty} p_m \frac{(x - x_0)^m}{m!}, \quad q(y) = \sum_{n=0}^{\infty} q_n \frac{(y - y_0)^n}{n!}. \end{array} \right.$$

We assume that $y(x_0) = h(y_0)$, $p(x_0) = q(y_0)$, and

$$(1.4) \quad |f_{m, n}| \leq M, |g_m| \leq M, |h_n| \leq M, |p_m| \leq M, |q_n| \leq M, (m, n = 0, 1, 2, \dots).$$

To solve the problem (1.1)–(1.2), we shall use the F -function approach of C. Truesdell, extending his F -equations to functions with several variables.

2. Following C. Truesdell [12], we call $F(x, y; \alpha, \beta)$ an „ F -function“, if

$$(2.1) \quad \frac{\partial F(x, y; \alpha, \beta)}{\partial x} = F(x, y; \alpha + 1, \beta), \quad \frac{\partial F(x, y; \alpha, \beta)}{\partial y} = F(x, y; \alpha, \beta + 1).$$

Suppose the function $\Phi(\alpha, \beta)$ is defined for $\alpha = \alpha_0 + m$, $\beta = \beta_0 + n$, ($m, n = 0, 1, 2, \dots$), where α_0, β_0 are constants, and is such that

$$(2.2) \quad |\Phi(\alpha_0 + m, \beta_0 + n)| \leq M_1, \quad (m, n = 0, 1, 2, \dots).$$

Then, by a straight-forward extension of an existence theorem due to C. Truesdell, we can easily show that the system of functional equations (2.1) admits a unique solution satisfying the condition

$$(2.3) \quad F(x_0, y_0; \alpha, \beta) = \Phi(\alpha, \beta), \quad \alpha = \alpha_0 + m, \quad \beta = \beta_0 + n, \quad (m, n = 0, 1, 2, \dots),$$

and this solution is given by the formula

$$(2.4) \quad F(x, y; \alpha, \beta) = \sum_{m, n=0}^{\infty} \Phi(\alpha + m, \beta + n) \frac{(x - x_0)^m}{m!} \frac{(y - y_0)^n}{n!}.$$

The function $F(x, y; \alpha, \beta)$ defined by (2.4) is an analytic function of exponential type¹⁾.

3. We now seek a solution to the problem (1.1), (1.2) in the form of an F -function. To do so, assume that this F -function say $F(x, y; \alpha, \beta)$, reduces to a function $\Phi(\alpha, \beta)$ for $(x, y) = (x_0, y_0)$. Hence, we have to de-

¹⁾ The concept of „ F -function“ can be extended to functions with more than two variables, using a similar way.

termine the function $\Phi(\alpha, \beta)$, defined for $\alpha = \alpha_0 + m$, $\beta = \beta_0 + n$, ($m, n = 0, 1, 2, \dots$), in such a way that the function $F(x, y; \alpha, \beta)$ given by the formula (2.4) will satisfy the equation (1.1) and the conditions (1.2).

Since $F(x, y; \alpha, \beta)$ is supposed to be an F -function, we have

$$(3.1) \quad \begin{cases} \frac{\partial^2}{\partial x \partial y} F(x, y; \alpha, \beta) = F(x, y; \alpha + 1, \beta + 1), \\ \frac{\partial^4}{\partial x^2 \partial y^2} F(x, y; \alpha, \beta) = F(x, y; \alpha + 2, \beta + 2) \end{cases}$$

by Eqs. (2.1). Thus, from (1.1)–(1.2), (2.4), and (3.1), we obtain the following system of partial difference equations

$$(3.2) \quad \begin{cases} F(x, y; \alpha + 2, \beta + 2) - \lambda F(x, y; \alpha, \beta) = f(x, y), \\ F(x, y_0; \alpha, \beta) = g(x), & F(x_0, y; \alpha, \beta) = h(y), \\ F(x, y_0; \alpha + 1, \beta + 1) = p(x), & F(x_0, y; \alpha + 1, \beta + 1) = q(y). \end{cases}$$

We now put

$$(3.3) \quad \Phi_{m,n} = \Phi(\alpha + m, \beta + n), \quad (m, n = 0, 1, 2, \dots).$$

By substituting the relevant power series on both sides of each of the equations in (3.2) and comparing the coefficients of similar terms, we immediately obtain the following recurrence relationships for the $\Phi_{m,n}$'s:

$$(3.4) \quad \begin{cases} \Phi_{m+2, n+2} - \lambda \Phi_{m,n} = f_{m,n}, & (m, n = 0, 1, 2, \dots), \\ \Phi_{m,0} = g_m, & \Phi_{0,n} = h_n, & \Phi_{m+1,1} = p_m, & \Phi_{1,n+1} = q_n. \end{cases}$$

Note that $g_0 = h_0$ and $p_0 = q_0$ by assumptions of § 1, and each $\Phi_{m,n} = \Phi(\alpha + m, \beta + n)$ is uniquely determined by the recurrent system (3.4), independently of the values of α and β . Further, starting with the inequalities (1.4), we can show by induction that

$$(3.5) \quad |\Phi_{m,n}| \leq M \sum_{k=0}^{r(m,n)} |\lambda|^k,$$

where, denoting by $[s]$ the integer part of s ,

$$(3.6) \quad r(m,n) = \min \left\{ \left[\frac{m}{2} \right], \left[\frac{n}{2} \right] \right\}.$$

Hence, for $|\lambda| < 1$, we have

$$(3.7) \quad |\Phi_{m,n}| \leq \frac{M}{1 - |\lambda|}, \quad (m, n = 0, 1, 2, \dots).$$

Thus, for any fixed (α_0, β_0) , the function $\Phi(\alpha, \beta)$, defined for $\alpha = \alpha_0 + m$, $\beta = \beta_0 + n$, $(m, n = 0, 1, 2, \dots)$ by (3.3), satisfies the condition (2.2). Consequently, the function $F(x, y; \alpha, \beta)$ given by formula (2.4) represents the unique F -function which is the unique analytic solution of the problem (1.1)–(1.2) for $|\lambda| < 1$.

4. *Remarks.* 1°. It is quite clear that the above method can be used in the solutions of the Darboux's type problems for more general polyvibrating equations. The novelty of our approach lies in the determination of the function $\Phi(\alpha, \beta)$ which generates the F -function by formula (2.4), in such a way that F satisfies the differential equation as well as the data on the characteristics.

2°. Further, in the above method, as observed by C. Truesdell we can use complex variables instead of real ones. For example, let z and ζ be complex variables ($z = x + iy$, $\zeta = \xi + i\eta$) and consider the polyvibrating equation

$$(4.1) \quad \frac{\partial^4 v(z, \zeta)}{\partial z^2 \partial \zeta^2} - \lambda v(z, \zeta) = f(z, \zeta)$$

subject to the conditions

$$(4.2) \quad \begin{aligned} v(z, \zeta_0) &= g(z), & v(z_0, \zeta) &= h(\zeta), \\ \frac{\partial^2 v(z, \zeta)}{\partial z \partial \zeta} \Big|_{\zeta=\zeta_0} &= p(z), & \frac{\partial^2 v(z, \zeta)}{\partial z \partial \zeta} \Big|_{z=z_0} &= q(\zeta), \end{aligned}$$

where f, g, h, p , and q are analytic functions given by power series similar to these which appear in (1.3). By the above method we can establish the unique analytic solution of the problem (4.1)–(4.2). Let $v = v(z, \zeta)$ be this solution. If we take $\zeta = \bar{z} = x - iy$ the polyvibrating equation (4.1) reduces to the biharmonic equation

$$(4.3) \quad \Delta \Delta v - \lambda v = f \quad \left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

In this way we find the solution of the biharmonic equation satisfying the conditions (4.2). The method of reduction of metaharmonic equations into polyvibrating equations with complex arguments has been used successively by I. N. Vekua and others.

3°. Since the data and the solution are of exponential type, the method of Laplace and Laplace-Picone transforms [13] can be applied in the solution of the problem (1.1)–(1.2) or (4.1)–(4.2).

4°. Finally, extending F -equation to functions with several variables, the authors prepared an essay toward a unified theory of special functions in the sense of C. Truesdell.

Received November 15, 1970

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PROBLEME REFERITOARE LA ECUAȚII POLIVIBRANTE

I. Soluții sub forma de F -funcții

(Rezumat)

Luînd ca punct de plecare un șir de rezultate proprii referitoare la ecuații polivibrante, numite de mai mulți oameni de știință, printre care și acad. G. Birkhoff, „ecuații Mangeron“ [8], autorii studiază soluția analitică a sistemului polivibrant (1.1)–(1.2). Metoda folosită constă în utilizarea punctului de vedere al lui C. Truesdell [12], convenabil extins la cazul mai multor variabile.

1. The first part of the report is a general introduction to the subject of the study. It discusses the importance of the study and the objectives of the research. It also provides a brief overview of the methodology used in the study.

2. The second part of the report is a detailed description of the study. It includes a description of the study area, the study population, and the study variables. It also includes a description of the data collection methods and the data analysis methods.

3. The third part of the report is a discussion of the results of the study. It discusses the findings of the study and their implications for the field of study. It also discusses the limitations of the study and the need for further research.

CORRESPONDENȚE PRIN HIPERPLANE TANGENTE PARȚIAL PARALELE ÎNTRE DOUĂ VARIETĂȚI RIGLATE DIN S_5

DE

ȘTEFANIA RUSCIOR

1. În această lucrare se consideră corespondențe prin varietăți liniare tangente parțial paralele între două varietăți riglate de dimensiune 3 și 2 dintr-un S_5 afin.

Se știe că D. Sommerville [1] a introdus noțiunea de paralelism parțial și paralelism complet în S_n euclidian. Plecînd de la aceste noțiuni cu ajutorul teoriei laticelor geometrice am considerat în [2], [3] cîteva aspecte algebrice ale corespondenței prin paralelism în spații afine. Relația de paralelism introdusă de Dubreil—Jacotin [4] nu se referă decît la paralelismul complet și nu este o relație simetrică. Astfel avem

$$A \parallel B \Rightarrow \begin{cases} A \cap B = \emptyset, \\ A \cup B \asymp A. \end{cases}$$

Dacă considerăm în S_3 un plan A și o dreaptă B , după definiția precedentă planul este paralel cu dreapta, dar dreapta nu este paralelă cu planul deoarece $A \cap B = \emptyset$ și $A \cup B = S_3$ care acoperă planul A dar nu acoperă dreapta B .

Într-o lucrare anterioară [5], am introdus într-un spațiu afin varietățile liniare mixte cu ajutorul cărora am definit relația de paralelism și dimensiunea paralelismului parțial.

Mulțimea varietăților liniare mixte M , aparținînd unui S_n afin, sînt varietăți liniare formate din puncte proprii și puncte improprii. Un punct propriu este caracterizat de n -upla $(x_1, x_2, \dots, x_n, 1)$ și un punct impropriu de $(x_1, x_2, \dots, x_n, 0)$.

În S_5 am considerat o latice geometrică proprie M de dimensiune 5 formată din varietăți liniare proprii $\emptyset, V_0, V_1, V_2, V_3, V_4, U$, și o latice geometrică impropriu M de dimensiune 4 formată din puncte improprii

$\emptyset, V_0^*, V_1^*, V_2^*, V_3^*, U$. O varietate mixtă $\bar{A} \in \bar{M}$ este definită de perechea (A, A^*) dacă:

$$\begin{cases} 1^\circ & A \in M, \quad A^* \in M^*, \\ 2^\circ & d[A] \text{ sau } d[A^*] \text{ este } \geq 1, \\ 3^\circ & d[A] - d[A^*] = \pm 1; \end{cases}$$

pentru perechea care conține elementul nul Φ avem:

$$\forall A \in M \Rightarrow (A, \emptyset) = (\emptyset, A) = A,$$

$$\forall A^* \in M^* \Rightarrow (A^*, \emptyset) = (\emptyset, A^*) = A^*.$$

Dimensiunea varietății liniare mixte $\bar{A}$ este un număr natural $d[\bar{A}] = \max \{d[A], d[A^*]\}$. Legea fundamentală care leagă dimensiunile a 2 varietăți mixte din S_5 este:

$$(1) \quad d[\bar{A}] + d[\bar{B}] \leq 5 + d[\bar{A} \cap \bar{B}].$$

Se arată că varietățile liniare mixte formează o semilattice geometrică (relativ la reuniune).

Definiție. Două varietăți liniare mixte distincte $\bar{A}$ și $\bar{B}$ de dimensiune ≥ 1 sînt numite paralele ($\bar{A} \parallel \bar{B}$) dacă intersecția lor este o varietate improprie C^* . Deci

$$\bar{A} \parallel \bar{B}, \text{ dacă } \bar{A} \cap \bar{B} = C^* \quad (C^* \in M^*).$$

Definiție. Se numește dimensiune δ a paralelismului pe mulțimea varietăților mixte $\bar{M}$ o funcție numerică $P(\bar{A}, \bar{B}) \geq 0$ definită pentru fiecare pereche $\bar{A} \parallel \bar{B}$ de dimensiune ≥ 1

$$\delta = P(\bar{A}, \bar{B}) = \frac{d(\bar{A} \cap \bar{B}) + 1}{\min \{d[\bar{A}], d[\bar{B}]\}}.$$

În S_5 dacă considerăm $\bar{V}_2$ și $\bar{V}_3$ avem:

$$V_2 \parallel V_3 \Rightarrow \bar{V}_2 \cap \bar{V}_3 = \begin{cases} V_1^* & \text{avem } \delta = \frac{2}{2} = 1, \\ V_0^* & \text{avem } \delta = \frac{1}{2}. \end{cases}$$

În primul caz varietățile liniare sînt complet paralele și au în comun o dreaptă improprie, în al doilea caz varietățile liniare sînt semiparalele și au ca intersecție un punct impropriu. În ambele cazuri legea fundamentală (1) este verificată, căci $2 + 3 \leq 5 + 1$, $2 + 3 \leq 5 + 0$.

Dacă considerăm două varietăți riglate cu 3 dimensiuni $\bar{V}_3$ și $\bar{W}_3$ paralele obținem trei cazuri:

$$\bar{V}_3 \parallel \bar{W}_3 \text{ dacă } \bar{V}_3 \cap \bar{W}_3 = \begin{cases} V_2^* & \text{avem } \delta = \frac{3}{3} = 1, \\ V_1^* & \text{avem } \delta = \frac{2}{3}, \\ V_0^* & \text{avem } \delta = \frac{1}{3}. \end{cases}$$

În primul caz varietățile liniare sînt complet paralele; ele admit ca intersecție un plan impropriu V_2^* . În al doilea caz varietățile liniare sînt paralele parțial de dimensiune $2/3$ și admit ca intersecție o dreaptă impropriu V_1^* . În al treilea caz varietățile liniare sînt parțial paralele de dimensiune $1/3$ și au ca intersecție un punct impropriu V_0^* .

În primele două forme de paralelism legea fundamentală [1] este verificată, deoarece evident $3 + 3 \leq 5 + 2$, $3 + 3 \leq 5 + 1$.

În cazul al treilea (1) nu este verificată. Prin urmare în S_5 avem numai varietăți liniare cu trei dimensiuni complet paralele și paralele parțial de dimensiune $2/3$.

Ne propunem acum să studiem:

a) o corespondență prin semiparalelism între două varietăți riglate $\bar{V}_2$ și $\bar{V}_3$ din S_5 și

b) o corespondență prin paralelism parțial de dimensiune $2/3$ între două varietăți riglate V_3 și $\bar{W}_3$ din S_5 .

2. Fie două varietăți riglate de dimensiune 2 și 3 de tipul (V_0^1) și (V_0^2) , adică varietăți riglate care nu au puncte singulare pe o generatoare generică. Vom spune că aceste varietăți riglate sînt puse în corespondență prin semiparalelism dacă în punctele corespunzătoare varietățile tangente duse respectiv la (V_2) și (V_3) sînt semiparalele. Fie (V_3) și (V_2) date de respectiv ecuațiile:

$$(2) \quad \bar{r} = \bar{\rho}(u, v) + w\bar{\alpha}(u, v), \quad \bar{r}^* = \bar{\rho}^*(u) + w^*\bar{\alpha}^*(u).$$

Hiperplanul tangent H dus într-un punct al generatoarei g de pe (V_3) este determinat de vectorii $\bar{\rho}_u + w\bar{\alpha}_u$, $\bar{\rho}_v + w\bar{\alpha}_v$, $\bar{\alpha}$, care formează o matrice de rang 3.

Planul tangent P dus într-un punct al generatoarei g de pe (V_2) este determinat de vectorii $\rho_u^* + w^*\alpha_u^*$, $\alpha^*(u)$.

H este semiparalel cu P dacă matricea funcțională

$$(\bar{\rho}_u + w\bar{\alpha}_u, \bar{\rho}_v + w\bar{\alpha}_v, \bar{\rho}_u^* + w^*\alpha_u^*, \bar{\alpha}, \bar{\alpha}^*)$$

are rangul 4, ceea ce implică ca determinantul corespunzător matricei să fie egal cu zero

$$(3) \quad [\bar{\rho}_u + w\bar{\alpha}_u, \bar{\rho}_v + w\bar{\alpha}_v, \bar{\rho}_u^* + w^*\bar{\alpha}_u^*, \bar{\alpha}, \bar{\alpha}^*] = 0.$$

Dezvoltînd acest determinant și ordonînd după coordonatele proiective w și w^* obținem corespondența căutată:

$$(4) \quad \begin{aligned} & ww^* \{[\bar{\alpha}_u, \bar{\rho}_v, \bar{\alpha}_u^*, \bar{\alpha}, \bar{\alpha}^*] + [\bar{\rho}_u, \bar{\alpha}_v, \bar{\alpha}_u^*, \bar{\alpha}, \bar{\alpha}^*]\} + \\ & + w \{[\bar{\alpha}_u, \bar{\rho}_v, \bar{\rho}_u^*, \bar{\alpha}, \bar{\alpha}^*] + [\bar{\rho}_u, \bar{\alpha}_v, \bar{\rho}_u^*, \bar{\alpha}, \bar{\alpha}^*]\} + \\ & + w^* \{[\bar{\rho}_u, \bar{\rho}_u^*, \bar{\alpha}_u^*, \bar{\alpha}, \bar{\alpha}^*] + [\bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_u^*, \bar{\alpha}, \bar{\alpha}^*]\} = 0. \end{aligned}$$

Relația (4) arată că între punctele a două generatoare corespunzătoare se stabilește cu ajutorul corespondenței prin varietăți tangente semiparalele, o corespondență omografică.

Considerînd o generatoare g^* pe (V_2) fixată prin $u = \text{const.}$ rezultă pe (V_3) o infinitate de generatoare corespunzătoare date de $u = \text{const.}$ și v arbitrar. Aceste generatoare formează o suprafață riglată (U) situată pe (V_3) . Toate generatoarele acestei suprafețe (U) corespund prin semiparalelism generatoarei g^* de pe (V_2) . Avem astfel o corespondență între generatoarele varietății (V_2) și suprafețele (U) de pe V_3 . Modul în care facem ca unei generatoare g^* să-i corespundă o generatoare g de pe (U) este arbitrar și poate fi determinat cu ajutorul unei funcții univoce, depinzînd de argumentul u . Fie acesta o funcție $v = \varphi(u)$ fixată; atunci unei generatoare g^* de pe (V_2) îi corespunde o generatoare determinată de pe suprafața U . Dar la toate generatoarele suprafeței U îi corespund aceeași generatoare g^* de pe (V_2) .

În consecință dacă fixăm un punct M^* de pe g^* dat de $w^* = \text{const.}$, punctul corespunzător M de pe generatoarea g este dat de valoarea lui w , determinat din omografia (4). Corespondența este astfel încît hiperplanul tangent dus în M la (V_3) este semiparalel cu planul tangent dus în M^* la (V_2) .

Se constată că dacă generatoarele corespunzătoare sînt paralele $\bar{\alpha} = \bar{\alpha}^*$, omografia (4) este nedeterminată și anume este identic verificată. În acest caz orice plan tangent dus într-un punct de pe g^* este semiparalel cu orice hiperplan dus într-un punct al generatoarei g . Am obținut următorul rezultat:

Două varietăți riglate (V_3) și (V_2) fiind respectiv de tipul (V_0^2) și (V_0^1) , pot fi puse în corespondență prin varietăți tangente semiparalele, dacă și numai dacă generatoarele corespunzătoare nu sînt paralele.

Unei generatoare g^* de pe (V_2) îi corespunde prin semiparalelism varietăților tangente o infinitate de generatoare (g) de pe (V_3) care formează o suprafață riglată (U) , corespunzătoare prin semiparalelism generatoarei g^* .

Suprafața (U) are proprietatea că orice hiperplan dus într-un punct al ei la (V_3) este semiparalel cu planul tangent dus varietății (V_2) într-un punct al generatoarei g^* .

Dacă se consideră o funcție univocă care să pună în corespondență generatoarea (g^*) cu o generatoare (g) de pe (U), atunci punctele corespunzătoare de pe cele două generatoare determină o omografie dată de relația (4).

3. Fie două varietăți riglate de dimensiune 3, V_3 și W_3 de tipul (V_0^2) și (W_0^2) adică varietăți riglate care nu admit puncte singulare pe o generatoare generică. Ecuațiile lor sînt:

$$(5) \quad \bar{r} = \bar{\rho}(u, v) + w\bar{\alpha}(u, v), \quad \bar{r}^* = \bar{\rho}^*(u, v) + w^*\bar{\alpha}^*(u, v).$$

Correspondența prin hiperplane tangente parțial paralele se stabilește dacă în punctele corespunzătoare hiperplanele tangente H și H^* duse la V_3 și respectiv la W_3 se intersectează după o dreaptă improprie.

Hiperplanul H este determinat de vectorii $\bar{r}_u, \bar{r}_v, \bar{\alpha}$ ale căror componente formează o matrice de rang 3. Hiperplanul H^* este determinat de vectorii $\bar{r}_u^*, \bar{r}_v^*, \bar{\alpha}^*$ care formează tot o matrice de rang 3. Cele două hiperplane H și H^* au un punct impropriu comun dacă $\bar{\alpha} = \bar{\alpha}^*$; ele au dreaptă comună improprie dacă determinantul

$$(6) \quad [\bar{r}_u, \bar{r}_v, \bar{r}_u^*, \bar{r}_v^*, \bar{\alpha}] = 0$$

este nul.

Înlocuind în (6) valorile lui $\bar{r}_u, \bar{r}_v, \bar{r}_u^*, \bar{r}_v^*$ determinate de (5) obținem:

$$(7) \quad [\bar{\rho}_u + w\bar{\alpha}_u, \bar{\rho}_v + w\bar{\alpha}_v, \bar{\rho}_u^* + w^*\bar{\alpha}_u^*, \bar{\rho}_v^* + w^*\bar{\alpha}_v^*, \bar{\alpha}] = 0.$$

Dezvoltînd acest determinant și ordonînd după coordonatele proiective w și w^* ale punctelor situate respectiv pe generatoarea corespunzătoare obținem relația:

$$(8) \quad Aw^2 + Bw^* + Cww^* + Mw + Nw^* + P = 0,$$

unde A, B, C, M, N, P sînt determinanți de ordinul cinci exprimați în funcție de elementele varietăților riglate V_3 și W_3 sub forma:

$$\left\{ \begin{aligned} A &= [\bar{\alpha}_u, \bar{\alpha}_v, \bar{\rho}_u^*, \bar{\rho}_v^*, \bar{\alpha}], \\ B &= [\bar{\alpha}_u, \bar{\alpha}_v, \bar{\rho}_u, \bar{\rho}_v, \bar{\alpha}], \\ C &= \{[\bar{\alpha}_u, \bar{\alpha}_v, \bar{\rho}_u, \bar{\rho}_v, \bar{\alpha}] + [\bar{\alpha}_u, \bar{\alpha}_v, \bar{\rho}_u, \bar{\rho}_u^*, \bar{\alpha}]\}, \\ M &= \{[\bar{\alpha}_u, \bar{\rho}_u^*, \bar{\rho}_v^*, \bar{\rho}_v, \bar{\alpha}] + [\bar{\alpha}_u, \bar{\rho}_u^*, \bar{\rho}_v^*, \bar{\rho}_u, \bar{\alpha}]\}, \\ N &= \{[\bar{\alpha}_u, \bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_v^*, \bar{\alpha}] + [\bar{\alpha}_u, \bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_u^*, \bar{\alpha}]\}, \\ P &= [\bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_u^*, \bar{\rho}_v^*, \bar{\alpha}]. \end{aligned} \right.$$

Relația (8) se poate exprima într-un sistem local convenabil ales cu baza $\bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_u^*, \bar{\alpha}_u, \alpha$.

Se constată că relația (8) se exprimă prin

$$(9) \quad (aw + bw^* + c)(a_1w + b_1w^* + c_1) = 0,$$

unde coeficienții a, b, c, a_1, b_1, c_1 sînt determinați în funcție de elementele varietăților V_3 și W_3 . Așa dar obținem rezultatul:

Correspondența prin hiperplane tangente parțial paralele de dimensiune $2/3$ stabilită între două varietăți riglate de dimensiune 3 de tip V_3^2 , cu generatoarele corespunzătoare paralele, este un produs de două omotetii date de (9).

Fixînd un punct g pe generatoarea generică $(g) \in V_3$ obținem două puncte pe generatoarea corespunzătoare $g^* \in W_3$ date de omotetiile conținute în (9), puncte în care hiperplanele tangente duse respectiv la V_3 și W_3 sînt parțial paralele de dimensiune $\delta = 2/3$.

Primită la 14 iulie 1970

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CORRESPONDANCE PAR HYPERPLANS PARTIELLEMENT TANGENTS ENTRE DEUX VARIÉTÉS RÉGLÉES DANS S_5

(Résumé)

On étudie la correspondance par variétés linéaires partiellement tangents entre deux variétés réglées de dimension 3 et 2, plongées dans un S_5 affín.

ALGEBRE ASOCIATIVE, NECOMUTATIVE CU SCALARI REALI, AL CĂROR CENTRU ESTE O ALGEBRĂ POLINOMIALĂ

DE

I. ENESCU

1. Ne propunem prezentarea acelor algebre care, avînd proprietățile indicate în titlu, pot avea drept centru o algebră polinomială peste același cîmp de bază.

Vom studia în acest scop cele patru clase posibile de algebre asociative simple, semisimple, nilpotente și cu radical propriu. Menționăm că nu ne va interesa dacă elementele centrale formează sau nu un ideal propriu ci vom urmări numai cazurile cînd centrul formează o subalgebră proprie. De aceea, cazurile în care singurul element central este zeroul algebrei vor fi eliminate din considerațiile ce vor fi făcute.

Vom nota prin: A — o algebră reală de ordin n , asociativă și necomutativă, C — ansamblul elementelor centrale ale lui A .

2. Fie A o algebră simplă. Atunci ea nu conține elemente propriu nilpotente dar poate avea elemente nilpotente.

Începem prin a observa că subalgebra centrală a lui A este proprie (întrucît există unitate principală) și ea nu conține nici un element nilpotent, pentru că în caz contrar ar fi propriu nilpotent pentru A . Într-adevăr, fie $x \in C$ astfel ca $x^\alpha = 0$ ($\alpha > 1$). Atunci, pentru orice element $a \in A$, au loc relațiile

$$(ax)^\alpha = (xa)^\alpha = x^\alpha a^\alpha = 0,$$

ceea ce implică existența unui radical propriu pentru A care nu ar fi simplă.

Acest fapt arată că subalgebra C însăși nu are radical și, cum nu poate fi nici nilpotentă (existența elementului unitate împiedică acest fapt), urmează că ea este semisimplă și atunci [1] este polinomială.

Observăm, în particular, că subalgebra centrală poate fi chiar simplă și atunci, evident, ea este izomorfă cu corpul de bază al scalarilor; este cazul unei algebre A simple normale.

3. În cazul când A este semisimplă, considerațiile precedente sînt evident valabile, în sensul că subalgebra centrală nu conține nici un element nilpotent, fiind deci semisimplă. Am stabilit astfel

Propoziția 1. *Centrul unei algebre A necomutative, asociative semisimplă este o subalgebră polinomială semisimplă, care nu poate fi simplă decît atunci cînd A este simplă normală.*

4. Fie A o algebră nilpotentă. Ținînd seama de faptul că o asemenea algebră nu conține nici un idempotent diferit de elementul nul, urmează

Propoziția 2. *Centrul unei algebre nilpotente nu poate fi o subalgebră polinomială.*

5. Considerăm acum o algebră A necomutativă, cu radical propriu, indecompozabilă și avînd unitate principală.

Distingem două cazuri:

1° C are radical propriu.

Posibilitatea existenței acestui caz rezultă din faptul că orice element nilpotent din C este propriu nilpotent pentru A .

Atunci [2] subalgebra centrală se prezintă sub forma unei sume (nedirecte) dintre un cîmp K și radicalul R al său: $C = K + R$.

Notînd prin respectiv $\beta = \text{ordin } K$ (evident K este considerat aici ca algebră peste corpul de bază al lui A) și $\delta = \text{ordin } R - \text{ordin } R^2$, urmează, utilizînd un rezultat din [1], că subalgebra centrală va fi sau nu polinomială după cum $\beta = \delta$ sau, respectiv $\beta \neq \delta$.

2° Centrul nu are radical propriu.

Atunci C este o algebră polinomială semisimplă.

6. În cazul cînd A avînd radical propriu și fiind indecompozabilă nu are unitate principală, sînt posibile trei situații și anume:

1° $C = \{0\}$,

2° C este nilpotentă,

3° C are unitate principală.

Se observă imediat că, în primele două cazuri, centrul nu poate fi o subalgebră polinomială.

În cazul 3° centrul poate fi o algebră semisimplă, deci polinomială, sau poate avea radical propriu, situație întîlnită deja.

Menționăm că, dacă C este semisimplă, ea poate fi și simplă, așa cum arată următorul exemplu:

	u_1	u_2	u_3
u_1	u_1	0	0
u_2	0	u_2	u_3
u_3	0	0	0

7. Cazurile prezentate epuizează problema pentru orice algebră indecompozabilă de ordin finit. Dacă A este o algebră decompozabilă, atunci chestiunea se reduce la studiul fiecărei componente din descompunerea în sumă directă.

Primită la 26 noiembrie 1970

Institutul politehnic, Iași,
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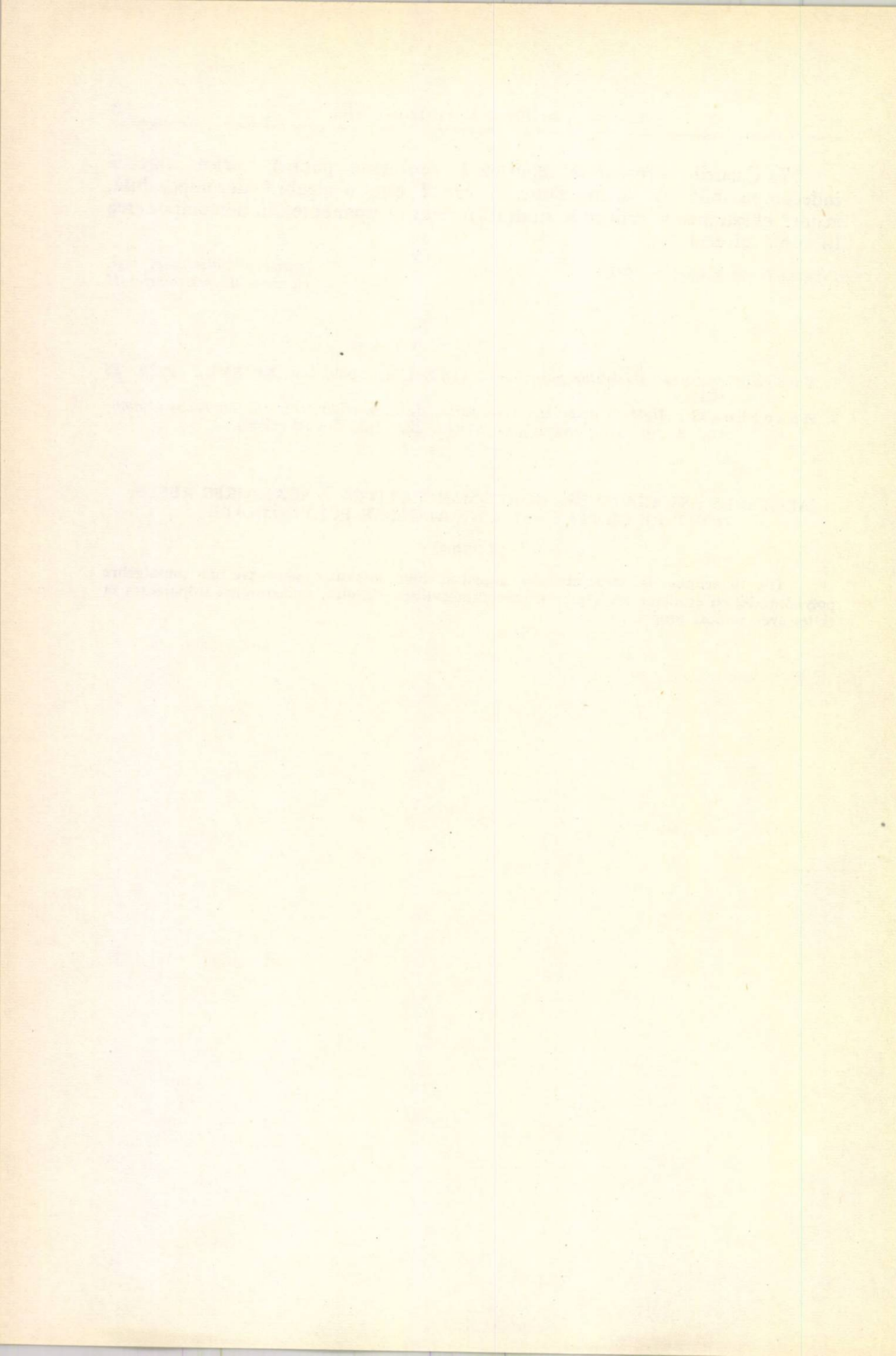
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ALGÈBRES ASSOCIATIVES, NONCOMMUTATIVES À SCALAIRES RÉELS, DONT LE CENTRE EST UNE ALGÈBRE POLYNÔMIALE

(Résumé)

On détermine la structure des algèbres dont le centre peut être une sousalgèbre polynômiale, en étudiant les algèbres noncommutatives, simples, semisimples, nilpotentes et celles avec radical propre.



SOME NEW ASPECTS CONCERNING KRILOFF AND SOBRERO FUNCTIONS ¹⁾

BY

ALFRED BRAIER

1. Exponential functions on algebras $\mathcal{A}_n$ on the real field $\mathcal{R}$ were introduced in [1], using their specific property of derivation ($f(xw) = \exp(xw)$, $w \in \mathcal{A}_n$, $x \in \mathcal{R}$, if for $\forall w$, $df(xw)/dx = wf(xw)$).

If we choose as basis in an associative and commutative algebra, with unit u , the powers of element v having as minimal polynomial the characteristic polynomial

$$(1) \quad P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n$$

of the ordinary differential equation with constant coefficients

$$(2) \quad \varphi^{(n)} + a_1\varphi^{(n-1)} + \dots + a_{n-1}\varphi' + a_n\varphi = 0,$$

then the components of $\exp(xw) = \sum_{i=1}^n \varphi_i(x)v_i$ in this basis are precisely fundamental functions of Eq. (2), satisfying the initial conditions

$$(3) \quad \left(\frac{d^p \varphi_k}{dx^p} \right)_{x=0} = \begin{cases} 1, & \text{if } p = k - 1 \\ 0, & \text{if } p \neq k - 1 \end{cases} \quad (k = 1, 2, \dots, n; p = 0, 1, \dots, n - 1).$$

In this paper we establish some new relations between the well known Kriloff, and also Sobrero functions, by utilising the main results of [1], just reminded.

2. Consider the equation

$$(4) \quad \varphi^{IV} + 4\varphi = 0;$$

its fundamental functions (3), are

¹⁾ Presented at the scientific session of the Polytechnic Institute of Jassy, July 2-4, 1970.

$$(5) \quad \begin{cases} \varphi(x) = \cos x \cosh x, & \psi(x) = \frac{1}{2} (\sin x \cosh x + \cos x \sinh x), \\ \theta(x) = \frac{1}{2} \sin x \sinh x, & \chi(x) = \frac{1}{4} (\sin x \cosh x - \cos x \sinh x), \end{cases}$$

known as Kriloff functions and largely utilised in the Strenght of Materials. Functions (5) are components of the exponential function

$$(6) \quad \exp(xv_2) = \varphi(x)v_1 + \psi(x)v_2 + \theta(x)v_3 + \chi(x)v_4,$$

in the cyclical basis $v_2^0 = v_1 (=u)$, $v_2^1 = v_2$, $v_2^2 = v_3$, $v_2^3 = v_4$ and (see (4), (1)) $v_2^4 = -4v_1$. The respective multiplication table is

$$(7) \quad \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & v_1 & v_2 & v_3 & v_4 \\ v_2 & v_2 & v_3 & v_4 & -4v_1 \\ v_3 & v_3 & v_4 & -4v_1 & -4v_2 \\ v_4 & v_4 & -4v_1 & -4v_2 & -4v_3 \end{array}$$

Taking into account that $\exp[(x+y)v_2] = \exp(xv_2) \exp(yv_2)$, it results $\varphi(x+y)v_1 + \psi(x+y)v_2 + \theta(x+y)v_3 + \chi(x+y)v_4 = [\varphi(x)v_1 + \psi(x)v_2 + \theta(x)v_3 + \chi(x)v_4][\varphi(y)v_1 + \psi(y)v_2 + \theta(y)v_3 + \chi(y)v_4]$. By performing the multiplications according to (7), we obtain the following functional relations:

$$(8) \quad \begin{cases} \varphi(x+y) = \varphi(x)\varphi(y) - 4\psi(x)\chi(y) - 4\theta(x)\theta(y) - 4\chi(x)\psi(y), \\ \psi(x+y) = \varphi(x)\psi(y) + \psi(x)\varphi(y) - 4\theta(x)\chi(y) - 4\chi(x)\theta(y), \\ \theta(x+y) = \varphi(x)\theta(y) + \psi(x)\psi(y) + \theta(x)\varphi(y) - 4\chi(x)\chi(y), \\ \chi(x+y) = \varphi(x)\chi(y) + \psi(x)\theta(y) + \theta(x)\psi(y) + \chi(x)\varphi(y). \end{cases}$$

We find that the same functions (5) occur also in the expression of the exponential function build up for element v_4 . By solving the respective system of equations, obtained from $d \exp\left(\frac{x}{2}v_4\right)/dx = \frac{v_4}{2} \exp\left(\frac{x}{2}v_4\right)$, it results that

$$(9) \quad \exp\left(\frac{x}{2}v_4\right) = \varphi(x)v_1 + 2\chi(x)v_2 - \theta(x)v_3 + \frac{1}{2}\psi(x)v_4.$$

Algebra (7) is the $\mathcal{P}$ -polynions algebra [2], cartesian product of two complex algebras. In the basis which made obvious this character, the multiplication table is

$$(10) \quad \begin{array}{c|cccc} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \hline \varepsilon_1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_2 & \varepsilon_2 & -\varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_3 & \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2 \\ \varepsilon_4 & \varepsilon_4 & -\varepsilon_3 & -\varepsilon_2 & \varepsilon_1 \end{array}$$

The necessary change of basis is

$$(11) \quad \varepsilon_1 = v_1, \quad \varepsilon_2 = \frac{v_2}{2} + \frac{v_4}{4}, \quad \varepsilon_3 = -\frac{v_3}{2}, \quad \varepsilon_4 = \frac{v_2}{2} - \frac{v_4}{4}.$$

However in basis (10)

$$(12) \quad \exp(x\varepsilon_2) = \cos x\varepsilon_1 + \sin x\varepsilon_2, \quad \exp(x\varepsilon_4) = \cosh x\varepsilon_1 + \sinh x\varepsilon_4.$$

Then we can write $\exp(2x\varepsilon_2) = \cos 2x\varepsilon_1 + \sin 2x\varepsilon_2 = \cos 2xv_1 + \frac{1}{2}\sin 2xv_2 + \frac{1}{4}\sin 2xv_4$; on the other hand $\exp(2x\varepsilon_2) = \exp(xv_2) \exp\left(\frac{x}{2}v_4\right)$. By dealing in the same manner with $\exp(2x\varepsilon_4)$, but having in mind that $\psi(x)$ and $\chi(x)$ are odd functions, we obtain

$$(13) \quad \begin{cases} \varphi^2(x) - 2\psi^2(x) + 4\theta^2(x) - 8\chi^2(x) = \cos 2x, \\ 2\varphi(x)\psi(x) - 4\psi(x)\theta(x) + 8\theta(x)\chi(x) + 4\chi(x)\varphi(x) = \sin 2x, \\ \varphi^2(x) + 2\psi^2(x) + 4\theta^2(x) + 8\chi^2(x) = \cosh 2x, \\ 2\varphi(x)\psi(x) + 4\psi(x)\theta(x) + 8\theta(x)\chi(x) - 4\chi(x)\varphi(x) = \sinh 2x. \end{cases}$$

Eqs. (13) results also in

$$(14) \quad \begin{cases} \varphi^2(x) + 4\theta^2(x) = \frac{1}{2}(\cosh 2x + \cos 2x), \\ \psi^2(x) + 4\chi^2(x) = \frac{1}{4}(\cosh 2x - \cos 2x), \\ \varphi(x)\psi(x) + 4\theta(x)\chi(x) = \frac{1}{4}(\sinh 2x + \sin 2x), \\ \psi(x)\theta(x) - \varphi(x)\chi(x) = \frac{1}{8}(\sinh 2x - \sin 2x), \end{cases}$$

which can be easily verified by means of direct calculation.

We start now from

$$(15) \quad \lambda = a\varepsilon_3 + b\varepsilon_4 = \frac{a+b}{2} v_2 + \frac{a-b}{2} \frac{v_4}{2}$$

and calculate $\exp \lambda = \exp(a\varepsilon_2) \exp(b\varepsilon_4) = \exp\left(\frac{a+b}{2} v_2\right) \exp\left(\frac{a-b}{2} \frac{v_2}{2}\right)$ taking into account (12), (10), (11) and (6), (9), (7). Finally we find other new relations, namely:

$$(16) \quad \left\{ \begin{array}{l} \sinh a \sin b + \sin a \sinh b = 4[\theta(m) \varphi(n) - \varphi(m) \theta(n)], \\ \sinh a \sin b - \sin a \sinh b = 8[\chi(m) \psi(n) - \psi(m) \chi(n)], \\ \cosh a \cos b + \cos a \cosh b = 2[\varphi(m) \varphi(n) + 4\theta(m) \theta(n)], \\ \cosh a \cos b - \cos a \cosh b = 4[\psi(m) \psi(n) + 4\chi(m) \chi(n)], \\ \sinh a \cos b + \cosh a \sin b = 2[-2\varphi(m) \chi(n) + \psi(m) \varphi(n) + 2\theta(m) \psi(n) + 4\chi(m) \theta(n)], \\ \sinh a \cos b - \cosh a \sin b = 2[\varphi(m) \psi(n) + 2\psi(m) \theta(n) + 4\theta(m) \chi(n) + 2\chi(m) \varphi(n)], \\ m = \frac{a+b}{2}, \quad n = \frac{a-b}{2}. \end{array} \right.$$

Let be the following functions on algebra (7):

$$(17) \quad C(xv) = \frac{1}{2} [\exp(xv) + \exp(-xv)], \quad S(xv) = \frac{1}{2} [\exp(xv) - \exp(-xv)].$$

Denoting $v_2 = v$ ($v_1 = v^0$, $v_3 = v^2$, $v_4 = v^3$) and utilising Eq. (6), (7) it results:

$$(18) \quad \left\{ \begin{array}{l} \varphi(x) v^0 = \frac{1}{2} \left[C(xv) + C\left(\frac{x}{2} v^3\right) \right], \quad \psi(x) v = \frac{1}{2} \left[S(xv) - \frac{v^2}{2} S\left(\frac{x}{2} v^3\right) \right], \\ \theta(x) v^2 = \frac{1}{2} \left[C(xv) - C\left(\frac{x}{2} v^3\right) \right], \quad \chi(x) v^3 = \frac{1}{2} \left[S(xv) + \frac{v^2}{2} S\left(\frac{x}{2} v^3\right) \right]. \end{array} \right.$$

The meaning of functions (17) has been explained in [3]; they represent trigonometric or hyperbolic functions on A_4 (7). Namely

$$(19) \quad \left\{ \begin{array}{l} C(xv) = \cosh(xv) = \cos\left(\frac{x}{2} v^3\right), \quad S(xv) = \sinh(xv) = -\frac{v^2}{2} \sin\left(\frac{x}{2} v^3\right), \\ C\left(\frac{x}{2} v^3\right) = \cosh\left(\frac{x}{2} v^3\right) = \cos(xv), \quad S\left(\frac{x}{2} v^3\right) = \sinh\left(\frac{x}{2} v^3\right) = \frac{v^2}{2} \sin(xv). \end{array} \right.$$

3. Consider now the equation

$$(20) \quad \varphi^{IV} + 2\varphi'' + \varphi = 0.$$

We obtain the well known multiplication table, in the cyclical basis of element v_2 , which characterises Sobrero algebra [4]:

$$(21) \quad \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & v_1 & v_2 & v_3 & v_4 \\ v & v_2 & v_3 & v_4 & -v_1 - 2v_3 \\ v_3 & v_3 & v_4 & -v_1 - 2v_3 & -v_2 - 2v_4 \\ v_4 & v_4 & -v_1 - 2v_3 & -v_2 - 2v_4 & -2v_1 + 3v_3 \end{array}.$$

Then

$$(22) \quad \begin{cases} \exp(xv_2) = \Phi(x)v_1 + \Psi(x)v_2 + \Theta(x)v_3 + \Gamma(x)v_4, \\ \exp(xv_3) = \lambda(x)v_1 + \mu(x)v_3, \\ \exp(xv_4) = \Phi^*(x)v_1 + \Psi^*(x)v_2 + \Theta^*(x)v_3 + \Gamma^*(x)v_4, \end{cases}$$

where

$$(23) \quad \begin{cases} \Phi(x) = \frac{1}{2}(2\cos x + x\sin x), & \Psi(x) = \frac{1}{2}(3\sin x - x\cos x), \\ \Theta(x) = \frac{1}{2}x\sin x, & \Gamma(x) = \frac{1}{2}(\sin x - x\cos x), \end{cases}$$

$$(24) \quad \lambda(x) = (1+x)e^{-x}, \quad \mu(x) = xe^{-x},$$

$$(25) \quad \begin{cases} \Phi^*(x) = \frac{1}{2}(2\cos x + 3x\sin x), & \Psi^*(x) = \frac{3}{2}(x\cos x - \sin x), \\ \Theta^*(x) = \frac{3}{2}x\sin x, & \Gamma^*(x) = \frac{1}{2}(3x\cos x - \sin x). \end{cases}$$

The functions (23) ... (25) will be called Sobrero functions. They have been introduced in [5, p. 13] in connection with problems of Elasticity. We can establish easily the following relations:

$$(26) \quad \begin{cases} \Phi(x+y) = \Phi(x)\Phi(y) - \Theta(x)\Theta(y) + 2\Gamma(x)\Gamma(y) - \Psi(x)\Gamma(y) - \Gamma(x)\Psi(y), \\ \Psi(x+y) = \Phi(x)\Psi(y) + \Psi(x)\Phi(y) - \Theta(x)\Gamma(y) - \Gamma(x)\Theta(y), \\ \Theta(x+y) = \Phi(x)\Theta(y) + \Theta(x)\Phi(y) + \Psi(x)\Psi(y) - 2\Theta(x)\Theta(y) + \\ \quad + 3\Gamma(x)\Gamma(y) - 2\Psi(x)\Gamma(y) - 2\Gamma(x)\Psi(y), \\ \Gamma(x+y) = \Phi(x)\Gamma(y) + \Gamma(x)\Phi(y) + \Psi(x)\Theta(y) + \Theta(x)\Psi(y) - \\ \quad - 2\Theta(x)\Gamma(y) - 2\Gamma(x)\Theta(y), \end{cases}$$

$$(27) \quad \begin{cases} \lambda(x+y) = \lambda(x)\lambda(y) - \mu(x)\mu(y), \\ \mu(x+y) = \lambda(x)\mu(y) + \mu(x)\lambda(y) - 2\mu(x)\mu(y). \end{cases}$$

The functional relations (26) hold also for functions (25).

The change of basis [2]

$$(28) \quad v_1 = \varepsilon_1, \quad v_2 = 2\varepsilon_2 + \varepsilon_3, \quad v_3 = -\varepsilon_1 + 4\varepsilon_4, \quad v_4 = -6\varepsilon_2 - \varepsilon_3$$

leads to the multiplication table

$$(29) \quad \begin{array}{c|cccc} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \hline \varepsilon_1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_2 & \varepsilon_2 & 0 & \varepsilon_4 & 0 \\ \varepsilon_3 & \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2 \\ \varepsilon_4 & \varepsilon_4 & 0 & -\varepsilon_2 & 0 \end{array},$$

which underlines that Sobrero algebra is the cartesian product of complex and dual algebras. Here

$$(30) \quad \begin{cases} \exp(x\varepsilon_2) = \varepsilon_1 + x\varepsilon_2, & \exp(x\varepsilon_3) = \cos x \varepsilon_1 + \sin x \varepsilon_3, \\ \exp(x\varepsilon_4) = \varepsilon_1 + x\varepsilon_4. \end{cases}$$

Starting from $\varepsilon_2 = -\frac{1}{4}(v_2 + v_4)$ and $\varepsilon_3 = \frac{1}{2}(3v_2 + v_4)$ and utilising

(30), (28) and (22), we obtain a series of relations between Sobrero functions (23), (25). Let be written only the following ones

$$(31) \quad \begin{cases} \Phi(x)\Phi^*(x) - \Theta(x)\Theta^*(x) + 2\Gamma(x)\Gamma^*(x) - \Psi(x)\Psi^*(x) - \Gamma(x)\Psi^*(x) = 1, \\ \Phi(x)\Psi^*(x) + \Psi(x)\Phi^*(x) - \Theta(x)\Gamma^*(x) - \Gamma(x)\Theta^*(x) = x, \\ \Phi(3x)\Phi^*(x) - \Theta(3x)\Theta^*(x) + 2\Gamma(3x)\Gamma^*(x) - \\ \quad - \Psi(3x)\Psi^*(x) - \Gamma(3x)\Psi^*(x) = \cos 2x, \\ \Phi(3x)\Psi^*(x) + \Psi(3x)\Phi^*(x) - \Theta(3x)\Gamma^*(x) - \Gamma(3x)\Theta^*(x) = \frac{3}{2}\sin 2x. \end{cases}$$

Received November 6, 1970

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UNELE ASPECTE INEDITE PRIVIND FUNCȚIILE KRÎLOV ȘI SOBRERO

(Rezumat)

Utilizând proprietăți ale funcției exponențiale construită pentru elementele bazei ciclice a unei algebre (comutative, asociative și cu unitate pe corpul real) [1], se deduc relațiile (8), (13), (14), (16) și (18) pe care le verifică cunoscutele funcții Krîlov (5). În mod similar se stabilesc și relațiile (26), (27) și (31) pentru funcțiile Sobrero (23)... (25).

QUASIVECTOR SPACES OVER FIELDS¹⁾

BY

DAN BORȘ

If X, Y are nonvoid sets, if T is a tribe of subsets of X which contains X and if T is a commutative group with respect to the symmetric difference and Y is a vector space over a field K , then [1] the set $T \times Y$ is a quasivector space over K with respect to the operations $(A, x) + (B, y) = (A \Delta B, x + y)$, $\alpha(A, x) = (A, \alpha x)$ where Δ is the symmetric difference, A and B are arbitrary sets in T , x and y are arbitrary elements in Y and α is an arbitrary scalar in K .

Definition 1. A nonvoid set X is called quasivector space over a field K if are defined the maps $(x, y) \rightarrow x + y$ of $X \times X$ in X , $(\alpha, x) \rightarrow \alpha x$ of $K \times X$ in X which satisfy the axioms:

1. For all x, y, z in X we have $(x + y) + z = x + (y + z)$.
2. For all x in X there is an element θ in X so that $x + \theta = \theta + x = x$.
3. For all x in X there is an element $-x$ in X so that $x + (-x) = (-x) + x = \theta$.
4. For all α in K and x, y in X we have $\alpha(x + y) = \alpha x + \alpha y$.
5. For all α, β in K and x in X we have $\alpha(\beta x) = (\alpha\beta)x$.
6. For all x in X we have $1x = x$.
7. For all α, β, γ in K and x in X we have $(\alpha + \beta + \gamma)x = \alpha x + \beta x + \gamma x$.

Examples. 1°. Any vector space over a field K is a quasivector space over K .

2°. Any periodic group G with all its elements of second order is a quasivector space over a field K if we define the external operation by $\alpha x = x$ for all α in K and x in G .

3°. The example of the quasivector space given in [1] permits a generalisation which is a method to construct quasivector spaces.

¹⁾ Presented at the scientific session of the Polytechnic Institute of Jassy, July 2-4, 1970.

If G is a group with respect to the operation o , V a vector space over a field K , then the set $\{e\} \times V$, e being the neutral element of G is a vector space of K with respect to the operations $(e, y) + (e, y') = (e, y + y')$, $\alpha(e, y) = (e, \alpha y)$ for all y, y' in V and α in K .

We prolong the operations of $\{e\} \times V$ to $G \times V$ so that $(x, y) + (x', y') = (xox', y + y')$, $\alpha(x, y) = (x, \alpha y)$ for all x, x' in G , y, y' in V and α in K .

Then the following conditions are equivalent:

(a) The set $G \times V$ is a quasivector space over K with respect to the operations written above.

(b) The set G is a periodic group with all its elements of second order.

Indeed, if the axiom 7 is satisfied then $x = xoxox$ for all x in G and conversely.

If $X = G \times V$ then $X_0 = \{e\} \times V$ is a vector space, $X^0 = G \times \{0\}$ is a quasivector space, $X_0 \cap X^0 = \{(e, 0)\}$ and $X = X_0 \oplus X^0$.

Consequences of the axioms 1...7.

1. The elements θ , $-x = (-1)x$ are uniquely determined.

2. The axiom 7 gives $\alpha x + \beta x = \beta x + \alpha x$ for all α, β in K and x in X .

3. The axioms 6 and 7 give $\theta = 0x + 0x$ or $0x = -0x$ for all x in X .

4. The equation $\alpha x = \theta$ has the solution $x = \theta$ because $\alpha\theta = \alpha(0x + 0x) = \theta$; this equation has also another solution without θ . If $\alpha \neq 0$ and $\alpha x = \theta$ we have $x = \theta$.

5. The axiom 7 is equivalent to $(\alpha + \beta)x = \alpha x + \beta x + 0x$ because, by consequence 2, we have $(\alpha + \beta + \gamma)x = (\alpha + \beta)x + \gamma x + 0x = \alpha x + \beta x + 0x + \gamma x + 0x = \alpha x + \beta x + \gamma x$.

6. For all $\alpha_1, \dots, \alpha_n$ in K and x in X we have

$$(\alpha_1 + \dots + \alpha_n)x = \begin{cases} \alpha_1 x + \dots + \alpha_n x + 0x, & \text{for } n \text{ even,} \\ \alpha_1 x + \dots + \alpha_n x, & \text{for } n \text{ odd.} \end{cases}$$

The relation is true for $n = 2$, $n = 3$ by consequence 7. If we assume that it is true for $n - 1$ we deduce that it is true also for n .

Definition 2. If X is a quasivector space over a field K the subset X' of X is called a quasivector subspace of X if X' is a quasivector space over K with respect to the restrictions to X' of the operations of X .

Proposition 1. If X is a quasivector space over a field K and if X' is a subset of X , the following conditions are equivalent:

(a) X' is a quasivector subspace of X ;

(b) for all x, y in X' , α in K we have $x + y$ in X' and αx in X' ;

(c) for all x, y in X' , α, β in K we have $\alpha x + \beta y$ in X' .

Proposition 2. The intersection of an arbitrary set of quasivector subspaces of a quasivector space X , is a quasivector subspace of X .

Definition 3. If X is a quasivector space over a field K , all quasivector subspaces X' of X which are vector spaces over K with respect to the restrictions to X' of the operations of X , are called banal quasivector subspaces; otherwise these spaces are called nonbanal quasivector subspaces of X .

Examples. 1° The subset $X_0 = \{x; x \text{ in } X, 0x = 0\}$ of X is a banal quasivector subspace of X .

2° The subset $X^\alpha = \{\alpha x; x \text{ in } X\}$ of X for α fixed in K is a nonbanal quasivector subspace of X . We have $X^1 = X$. Therefore $X^0 = \{0x; x \text{ in } X\}$ is a quasivector subspace of X which contains 0 and all the elements of the form $0x \neq 0$ for all x in X . The set X^0 is a periodic group with all its elements of second order with respect to the operation of group and $\alpha 0x = 0x$ for all α in K and x in X .

Remark. We shall denote by (X, X^0, X_0) a quasivector space over a field K .

3° If X' is a quasivector subspace of (X, X^0, X_0) then $X'^0 = X^0 \cap X'$, $X'_0 = X_0 \cap X'$ are quasivector subspaces of X ; X'^0 is a nonbanal and X'_0 is a banal quasivector subspace. We have $X'^0 \subset X^0$ and $X'_0 \subset X_0$. If X'' is another quasivector subspace of X , then $(X' \cap X'')^0 = X'^0 \cap X''^0$ and $(X' \cap X'')_0 = X'_0 \cap X''_0$.

Proposition 3. If (X, X^0, X_0) is a quasivector space over a field K and if X' is a quasivector subspace of X , then:

(a) $X'^0 \cap X'_0 = \{0\}$,

(b) the conditions $X' = X'_0$, $X'^0 = \{0\}$ are equivalent.

Therefore $X^0 \cap X_0 = \{0\}$ and $X = X_0$, $X^0 = \{0\}$ are equivalent.

Proposition 4. If (X, X^0, X_0) is a quasivector space over a field K , any banal quasivector subspace of X is a vector subspace of X_0 .

Proposition 5. If (X, X^0, X_0) is a quasivector space over a field K , the following conditions are equivalent:

(a) the group X is commutative;

(b) $X + X^0 = X^0 + X$.

Indeed, from identity $(1+1)(x+y) = (1+1)(x+y)$ we obtain $x+y+x+y+0x+0y = x+x+0x+y+y+0y$ or, by (b), $(y+x)+y = (x+y)+y$ therefore $y+x = x+y$.

Remark 1. For $G \times V$ (example 3) the equivalent conditions of proposition 5 means that G is commutative.

Definition 4. The quasivector space (X, X^0, X_0) over a field K is called commutative if are satisfied the equivalent conditions of proposition 5.

Remark 2. All banal quasivector subspaces of a quasivector space are commutative; all quasivector subspaces of a commutative quasivector space are commutative.

Proposition 6. Let (X, X^0, X_0) be a quasivector space over a field K , X' a quasivector subspace of X , $M(\alpha, \beta) = \{\alpha x + \beta x; x \text{ in } X'\}$ a subset of X for α, β fixed in K . Then:

(a) the sets X' and $M(1, 0)$ are equal;

(b) if X is commutative, $M(\alpha, \beta)$ is a banal quasivector subspace of X included in X'_0 ; moreover, in this case, the quasivector spaces X' and $M(1, 0)$ are equal.

Indeed, $0(\alpha x + \beta x) = 0$ and $\alpha x + \beta x + \alpha y + \beta y = \alpha(x + y) + \beta(x + y)$ only if X is commutative.

Remark 3. According to the proposition 6, for all x in X' we have uniquely $x = 0x + x + 0x$ with $0x$ in X'^0 and $x + 0x$ in X'_0 .

Proposition 7. If X', X'' are quasivector subspaces of the quasivector space (X, X^0, X_0) over a field K , then the set $X' + X''$ is a quasivector subspace of X with respect to the operations $(x' + x'') + (y' + y'') = (x' + y') + (x'' + y'')$, $\alpha(x' + x'') = \alpha x' + \alpha x''$ for all x', y' in X' , x'', y'' in X'' and α in K ; moreover $(X' + X'')^0 = X'^0 + X''^0$ and $(X' + X'')_0 = X'_0 + X''_0$.

Proposition 8. If X' is a quasivector subspace of a commutative quasivector space (X, X^0, X_0) over a field K , then $X' = X'^0 + X'_0$. Therefore $X = X^0 + X_0$.

Proposition 9. If (X, X^0, X_0) is a quasivector space over a field K and X_1, X_2 there are quasivector subspaces of X , the following conditions are equivalent:

(a) $X_1 \cap X_2 = X_1^0 \cap X_2^0$;

(b) for all $x = x_1 + x_2$ in $X_1 + X_2$, if $x = x'_1 + x'_2$, x'_1 in X_1 , x'_2 in X_2 then there exists y in $X_1^0 \cap X_2^0$ so that $x'_1 = y + x_1$, $x'_2 = y + x_2$.

Indeed, from $x_1 + x_2 = x'_1 + x'_2$ we deduce $-x'_1 + x_1 = x'_2 - x_2$ in $X_1^0 \cap X_2^0$, therefore there exists y in $X_1^0 \cap X_2^0$ so that $-x'_1 + x_1 = x'_2 - x_2 = y$. Conversely, if z is in $X_1 \cap X_2$ we have $z = z + 0$, z in X_1 , 0 in X_2 and $z = 0 + z$, 0 in X_1 , z in X_2 ; according to (b) there exists y in $X_1^0 \cap X_2^0$ so that $0 = y + z$, $z = y + 0$ or $z = y$.

Definition 5. If X is a quasivector space over a field K , X_1 and X_2 there are quasivector subspaces of X , then the quasivector subspace $X' = X_1 + X_2$ is called quasidirect summ of subspaces X_1 and X_2 if one of equivalent conditions of proposition 9 is satisfied. In this case we write $X' = X_1 \oplus X_2$. If at least one of quasivector subspaces X_1, X_2 of proposition 9 is banal then $X_1^0 \cap X_2^0 = \{0\}$ and the decomposition $x = x_1 + x_2$ is uniquely determined; in this case we call direct the summ $X_1 + X_2$.

Theorem 1. If (X, X^0, X_0) is a commutative quasivector space over a field K and if X' is a quasivector subspace of X then $X' = X'^0 \oplus X'_0$, this summ being direct. Therefore $X = X^0 \oplus X_0$.

Proposition 10. *If (X, X^0, X_0) , (Y, Y^0, Y_0) are quasivector spaces over a field K , the set $X \times Y$ is a quasivector space over K with respect to the operations*

$$(x, y) + (x', y') = (x + x', y + y'), \quad \alpha(x, y) = (\alpha x, \alpha y)$$

for all x, x' in X, y, y' in Y, α in K ; moreover $(X \times Y)^0 = X^0 \times Y^0$, $(X \times Y)_0 = X_0 \times Y_0$.

If X, Y are commutative quasivector spaces then $X \times Y$ is commutative and $X \times Y = (X \times \{0_y\}) \oplus (\{0_x\} \times Y)$, the summ being direct.

Definiton 6. *If X, Y are quasivector spaces over a field K , any bijection $f: X \rightarrow Y$ which satisfies the properties $f(x + x') = f(x) + f(x')$, $f(\alpha x) = \alpha f(x)$ for all x, x' in X and α in K , is called an isomorphism of quasivector space structures defined on the sets X and Y .*

Proposition 11. *If X, X' are sets with the same cardinal so that there exists a bijection $f: X \rightarrow X'$ and if X is a quasivector space over a field K with respect to the operations $(x, y) \rightarrow x + y$, $(\alpha, x) \rightarrow \alpha x$ then X' is a quasivector space over K with respect to the operations*

$$(x', y') \rightarrow x' + y' = f(f^{-1}(x') + f^{-1}(y')),$$

$$(\alpha, x') \rightarrow \alpha x' = f(\alpha f^{-1}(x'))$$

which is isomorphic with X .

Indeed, the neutral element of X' is $f(0)$ if 0 is the neutral element of X , the element $f(-f^{-1}(x'))$ is the symmetrical element of x' in X' ; for example, the axiom 7 is satisfied because

$$(\alpha + \beta)x' = f((\alpha + \beta)f^{-1}(x')) = f(\alpha f^{-1}(x') + \beta f^{-1}(x') + 0f^{-1}(x')),$$

$$\alpha x' + \beta x' + 0x' = f(f^{-1}(\alpha x') + f^{-1}(\beta x') + f^{-1}(0x')) =$$

$$= f[f^{-1}(f(\alpha f^{-1}(x')))) + f^{-1}(f(\beta f^{-1}(x')))) + f^{-1}(f(0f^{-1}(x')))) =$$

$$= f(\alpha f^{-1}(x') + \beta f^{-1}(x') + 0f^{-1}(x')).$$

The relations $f^{-1}(x' + y') = f^{-1}(x') + f^{-1}(y')$, $f^{-1}(\alpha x') = \alpha f^{-1}(x')$ means that $f^{-1}: X' \rightarrow X$ is an isomorphism, therefore f is an isomorphism.

Theorem 2. (a) *If X is a commutative quasivector space over a field K there exists a comutative periodic group G with all its elements of second order and a vector space V over K so that X is isomorphic to $G \times V$.*

(b) *If G is a commutative periodic group G with all its elements of second order and if V is a vector space over a field K , there exists a commutative quasivector space X over K so that $G \times V$ is isomorphic to X .*

Proof. (a) Let (X, X^0, X_0) be a commutative quasivector space over a field K ; then $X = X^0 \oplus X_0$ (theorem 1). We define the map $f: X \rightarrow X^0 \times X_0$ by $f(x) = (0x, x_0)$ for all $x = 0x + x_0$ in X , where x_0 in X_0 is uniquely determined and depends of x (theorem 1). By definition f is

surjective; if $x_1 = 0x_1 + x_{10}$, $x_2 = 0x_2 + x_{20}$, $f(x_1) = f(x_2)$ then $(0x_1, x_{10}) = (0x_2, x_{20})$ or $0x_1 = 0x_2$ and $x_{10} = x_{20}$, therefore $x_1 = x_2$ and f is injective. Moreover, if $x_1 = 0x_1 + x_{10}$, $x_2 = 0x_2 + x_{20}$, $x = 0x + x_0$ we have $f(x_1 + x_2) = (0(x_1 + x_2), (x_1 + x_2)_0) = (0x_1 + 0x_2, x_{10} + x_{20}) = f(x_1) + f(x_2)$, $f(\alpha x) = (0(\alpha x), (\alpha x)_0) = (\alpha(0x), \alpha x_0) = \alpha f(x)$. Therefore f is an isomorphism.

(b) Let M be an arbitrary set and f an injection of the quasivector space $G \times V$ in M ; then the map $f: G \times V \rightarrow f(G \times V)$ is bijective and the set $X = f(G \times V)$ is a quasivector space over K with respect to the operations defined, by means of f , in proposition 11, which is isomorphic to $G \times V$. Q.e.d.

Received August 1, 1970

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SPAȚII CUASIVECTORIALE PESTE CÎMPURI

(Rezumat)

Un spațiu cuasivectorial peste un cîmp K este o mulțime nevidă pentru care se definește o adunare și o înmulțire cu scalari care satisfac axiomele 1...7. Un astfel de spațiu are elemente de forma $0x$ diferite de 0 . Un spațiu cuasivectorial comutativ este sumă directă a subspațiilor X^0 și X_0 , izomorf cu $G \times V$ unde G este un grup periodic pentru care toate elementele sînt de ordinul al doilea și V un spațiu vectorial peste cîmpul K , reciproca fiind adevărată.

CATÉGORIES SCINDÉES ASOCIÉES AUX PRÉFAISCEAUX D'ENSEMBLES

PAR

GH. RADU

Notons par U un univers Grothendieck de sorte que toutes les catégories considérées seront U petites.

Soit E une U -petite catégorie et $P: E^{\circ} \rightarrow \text{Ens}$ un préfaisceau d'ensembles. Si l'on considère chaque ensemble, objet de $U\text{-Ens}$ comme une U -petite catégorie discrète, alors P peut être interprété comme étant un préfaisceau $P: E^{\circ} \rightarrow \text{Cat}$.

On prouvera que E -catégorie scindée associée à P coïncide avec la catégorie E/P et le foncteur de projection est le foncteur source généralisé $s_P: E/P \rightarrow E$.

En effet, si $P: E^{\circ} \rightarrow \text{Cat}$ est le foncteur construit ci-dessus, alors une E -catégorie scindée, soit-elle F , est définie comme ayant pour objet les éléments de l'ensemble $\coprod_{S \in \text{Ob } E} \text{Ob } P(S)$.

Un objet de F est donc, une paire $\xi = (\xi, S), \xi \in \text{Ob } P(S)$.

Conformément au lemme de Yoneda il existe un morphisme fonctoriel $\varphi^{\xi}: h_S \rightarrow P$ et toujours en vertu de ce lemme il résulte qu'un objet de F est bien déterminé par un morphisme $\varphi: h_S \rightarrow P$. Soit $\eta = (\eta, T)$ un autre objet de F et soit $\eta^{\eta}: h_T \rightarrow P$ le morphisme fondamental de la classe η , c'est-à-dire celui associé à $\eta \in P(T)$ par le lemme de Yoneda. Si $u: \eta \rightarrow \xi$ est un morphisme de F , il résulte l'existence d'un morphisme dans E , $f: T \rightarrow S$ et d'un morphisme $u: \eta \rightarrow P(f)(\xi)$ de $P(T)$.

De plus, on a le diagramme commutatif

(*)

$$\begin{array}{ccc}
 h_T & \xrightarrow{\varphi^{\eta}} & P \\
 \downarrow h_f & \searrow \varphi^{\xi} & \\
 h_S & &
 \end{array}$$

En effet $\varphi^\xi \cdot h_f(1_T) = \varphi^\xi(f) = P(f)(\xi) = \eta$ et le caractère commutatif du diagramme (*) résulte maintenant conformément au lemme de Yoneda. Il est évident que, les applications

$$\bar{\eta} = (\eta, T) \sim \sim \sim \rightarrow (\varphi^\eta, T)$$

$$u: \bar{\eta} \longrightarrow \bar{\xi} \sim \sim \sim \rightarrow f: (\varphi^\eta, T) \longrightarrow (\varphi^\xi, S)$$

définissent un foncteur de la catégorie F à la catégorie E/P qui représente un isomorphisme de catégories.

Proposition 1. Soit $u: E' \rightarrow E$ un foncteur covariant pleinement fidèle et $P: E^\circ \rightarrow U\text{-Ens}$ un préfaisceau d'ensembles sur E . Alors, E' -catégorie scindée associée au foncteur $P \circ u: E'^\circ \rightarrow \text{Ens}$ est la catégorie qu'on obtient de

E -catégorie scindée $E/P \xrightarrow{s_P} E$ par le changement de base $u: E' \rightarrow E$, c'est-à-dire on donne le diagramme cartésien de Cat

$$\begin{array}{ccc} E/P \times E' & \longrightarrow & E' \\ \downarrow E & & \downarrow u \\ E/P & \xrightarrow{s_P} & E \end{array}$$

Preuve. Par définition, $E'/P \circ u = E' \wedge_{E'} / P \circ u \times E'$. Soit (X, α) un objet de $E'/P \circ u$, c'est-à-dire $X \in \text{Ob}(E')$, $\alpha: h_x \rightarrow P \circ u$ et $x = \alpha_X(1_X) \in P(u(X))$. Grâce au lemme de Yoneda il existe un morphisme fonctoriel $\varphi^x: h_{u(X)} \xrightarrow{f} P$ qui définit un objet de $E/P \times_{E'} E'$, $((u(X), \varphi^x), X)$. Si $(X, \alpha) \xrightarrow{f} (Y, \beta)$ est un morphisme dans $E'/P \circ u$ alors, le morphisme $u(f): u(X) \rightarrow u(Y)$ définit un morphisme dans E/P car on a le diagramme commutatif

$$(D) \quad \begin{array}{ccc} h_{u(X)} & \xrightarrow{\varphi^x} & P \\ \downarrow h_{u(f)} & \searrow \varphi^y & \\ h_{u(Y)} & & \end{array}$$

En effet $\varphi_{u(X)}^{(x)}(1_{u(X)}) = \alpha_X(1_X) = x$, $\varphi_{u(X)}^y \circ h_{u(f)}(1_{u(X)}) = P(u(f))(\beta_Y(1_Y)) = (P \circ u)(f)(\beta_Y(1_Y)) = \alpha_X(1_X) = x$ et maintenant le caractère commutatif du diagramme (D) résulte avec le lemme de Yoneda.

Si u est pleinement fidèle, alors on a le foncteur

$$\lambda: E'/P \circ u \longrightarrow E/P \times_{E'} E',$$

qui est pleinement fidèle, car si on a le diagramme (D) commutatif il en résulte qu'on a aussi le diagramme commutatif

$$(D') \quad \begin{array}{ccc} h_X & \xrightarrow{\alpha} & P \circ u \\ \downarrow h_f & \searrow \beta & \\ h_Y & & \end{array}$$

En effet, dans le diagramme (D'), α et β sont des morphismes de classe fondamentale $\xi = \varphi_{u(X)}^x(1_{u(X)})$ respectivement $\eta = \varphi_{u(Y)}^y(1_{u(Y)})$ définis par le lemme de Yoneda et on a successivement

$$\beta \circ h_f(1_X) = \beta_X(f) = (P \circ u)(f)(\beta_Y(1_Y)) = \alpha_X(1_X) = x.$$

Le caractère commutatif du diagramme (D') résulte maintenant du lemme de Yoneda.

Enfin si $((X, \alpha), X')$ est un objet de $E/P \times_{E'} E'$ c'est-à-dire $\alpha: h_X \rightarrow P$ et $u(X') = X$, alors ça prouve qu'on a un morphisme $\alpha: h_{u(X')} \rightarrow P$ et conformément au lemme de Yoneda il en résulte qu'il y a un morphisme $\alpha_1: h_{X'} \rightarrow P \circ u$ et (X', α_1) est un objet de $E'/P \circ u$ qui par λ est porté dans l'objet $((X, \alpha), X')$. La proposition est ainsi prouvée.

Proposition 2. Soit E et E' de x U_0 -petites catégories et soit dans Cat le diagramme $E \xrightleftharpoons[v]{u} E'$ où v est un adjoint à gauche de u . Soit $u_*: E' \wedge \rightarrow E^*$ le foncteur image directe de préfaisceaux et $u^*: E' \wedge \rightarrow E' \wedge$ le foncteur image inverse de préfaisceaux.

Si P est un objet de $E' \wedge$ et $E/P, E'/u^*(P)$ les catégories scindées associées aux préfaisceaux P et $u^*(P)$, alors le foncteur canonique $\bar{u}: E/P \rightarrow E'/u^*(P)$ admet un adjoint à gauche $\bar{v}$.

Preuve. Soit (X, α) un objet de $E/P, \alpha: h_X \rightarrow P$ un morphisme fonctoriel. En appliquant à ce morphisme le foncteur u^* et en tenant compte que u^* rend commutatif le diagramme

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ h \downarrow & & \downarrow h' \\ E^\wedge & \xrightarrow{u^*} & E'^\wedge \end{array}$$

il en résulte qu'il existe un morphisme fonctoriel $u^*(\alpha): h'_{u(X)} \rightarrow u^*(P)$.

On définit $\bar{u}(X, \alpha) = (u(X), u^*(\alpha))$; on obtient l'action de $\bar{u}^*$ sur les morphismes toujours en appliquant le foncteur u^* .

Soit (X', α') un objet de $E'/u^*(P)$, c'est-à-dire $\alpha': h_{X'} \rightarrow u^*(P)$. Par le foncteur v^* ce morphisme devient $\alpha = v^*(\alpha'): h_{v(X')} \rightarrow v^*u^*(P)$ et puisque $v_* \approx u^*$, il existe jusqu'à un isomorphisme le morphisme fonctoriel, $\alpha: h_{v(X')} \rightarrow v^*v_*(P)$.

Si $\Psi_P: v^*v_*(P)$ est le composant d'indice P de la flèche d'adjonction Ψ , alors le morphisme composé $\Psi_P \circ v^*(\alpha'): h_{v(X')} \rightarrow P$ définit un objet de la catégorie E/P .

On va définir $\bar{v}(X', \alpha') = (v(X'), \Psi_P \circ v^*(\alpha'))$. Soit $f: (X', \alpha') \rightarrow (Y', \beta')$ un morphisme dans la catégorie $E'/u^*(P)$, donné par le diagramme commu-

$$\text{tatif, } \begin{array}{ccc} h_{X'} & \xrightarrow{\alpha'} & u^*(P) \\ h_{f'} \downarrow & & \uparrow h_{Y'} \\ h_{Y'} & \xrightarrow{\beta'} & \end{array}$$

Si l'on applique le foncteur v^* au diagramme (D) et si l'on compose ensuite à gauche avec Ψ , il en résulte que $v(f') = f: v(X') \rightarrow v(Y')$ définit dans E/P un morphisme. Le foncteur $\bar{v}$ est ainsi, défini.

Soit (X, α) un objet de E/P , $\alpha: h_X \rightarrow P$ un morphisme fonctoriel qui par u^* est devenu $u^*(\alpha): h_{u(X)} = u^*(P)$. En appliquant à ce-dernier le foncteur $\bar{v}$ on obtient le morphisme fonctoriel

$$\Psi_P \circ v^*u^*(\alpha) \approx \Psi v^*v_*(\alpha): h_{vu(X)} \rightarrow P.$$

Si Ψ_X est la flèche d'adjonction d'indice X pour la paire (u, v) on prouvera que Ψ_X définit un morphisme, $\bar{\Psi}(X): \bar{v}u(X, \alpha) \rightarrow (X, \alpha)$.

En effet, le diagramme

$$\begin{array}{ccc} v^*u^*h_X \approx h_{vu(X)} & \xrightarrow{\Psi_P v^*u^*(\alpha)} & P \\ \downarrow h_{\Psi(X)} & \nearrow \alpha & \\ h_X & & \end{array}$$

est commutatif, car Ψ étant flèche d'adjonction pour la paire v^*v_* on a le diagramme commutatif

$$\begin{array}{ccc} v^*v_*(h_X) & \xrightarrow{\Psi_{h_X}} & h_X \\ \downarrow v^*v_*(\alpha) & \Psi_P & \downarrow \alpha \\ v^*v_*(P) & \xrightarrow{\quad} & P \end{array}$$

On va prouver que $\bar{v}$ est un adjoint à gauche de $\bar{u}$ si $\bar{\Psi}(X)$ induit une application bijective

$$\text{Hom}_{E'/u^*(P)}((X', \alpha'), \bar{u}(X, \alpha)) \rightarrow \text{Hom}_{E/P}(\bar{v}(X', \alpha'), (X, \alpha)); \quad f \sim \rightarrow \bar{\Psi}(X)\bar{v}(f).$$

Si $f_1, f_2: (X', \alpha') \rightarrow \bar{u}(X, \alpha)$, $f_1 \neq f_2$ alors,

$$\Psi(X)v(f_1) \neq \Psi(X)v(f_2)$$

sinon, on aurait $\Psi(X)v(f_1) = \Psi(X)v(f_2)$. Mais (u, v) étant une paire d'adjonction on a l'application bijective

$$\text{Hom}_{E'}(X', u(X)) \longrightarrow \text{Hom}_E(v(X'), X)$$

$$f_1 \sim \sim \sim \rightarrow \Psi_X \circ u(f_1),$$

ce qui conduit à $f_1 = f_2$ et donc $f_1 = f_2$. On a prouvé ainsi que φ est une application injective.

Soit maintenant, $g: u(X', \alpha') \rightarrow (X', \alpha)$ donné par le diagramme comutatif

$$\begin{array}{ccc} h_{v(X')} & \xrightarrow{\Psi_P \circ v^*(\alpha')} & P \\ \downarrow h_g & \searrow \alpha & \\ h_X & & \end{array} \quad g: v(X') \rightarrow X.$$

Grâce à la paire d'adjonction (u, v) il en résulte que g existe uniquement si $X' \rightarrow u(X)$ de sorte que $g = \Psi_X \circ v(g_1)$.

Ou prouvera qu'on a aussi le diagramme commutatif

$$(D_1) \quad \begin{array}{ccc} h_{X'} & \xrightarrow{\alpha'} & u^*(P) \\ \downarrow h_{g_1} & \searrow u^*(\alpha) & \\ h_{u(X)} & & \end{array}$$

Puisque (v_*, v^*) est une paire d'adjonction il en résulte qu'on a une application bijective

$$\text{Hom}_{E'}(h_{X'}, v_*(P)) \xrightarrow{\sim} \text{Hom}_E(v^*(h_{X'}), P)$$

$$g_1: h_{X'} \rightarrow v_*(P) \sim \sim \sim \rightarrow v^*(h_{X'}) \xrightarrow{\Psi_P \circ v^*(g_1)} P.$$

Pour prouver que le diagramme (D_1) est commutatif il suffit de montrer qu'on a

$$\Psi_P \circ v^*(\alpha') = \Psi_P \circ v^*(u^*(\alpha)) \circ v^*(h_{g_1}).$$

Mais $\Psi_P v^*(\alpha') = \alpha h_g = \alpha h_{\Psi_X \circ h_{v(g_1)}} = \Psi_P v^* u^*(\alpha) v^*(h_{g_1})$, car le morphisme $\alpha: h_X \rightarrow P$ conduit au diagramme commutatif

$$\begin{array}{ccc}
 v^*u^*(h_X) & \xrightarrow{\Psi_{h_X}} & h_X \\
 \downarrow v^*u^*(\alpha) & & \downarrow \alpha \\
 v^*u^*(P) & \xrightarrow{\Psi_P} & P
 \end{array}$$

Le diagramme (D_1) définit donc un morphisme

$$\dot{g}_1 : (X', \alpha') \rightarrow (u(X), u^*(\alpha))$$

dans $E'/u^*(P)$. Enfin, $\varphi(\dot{g}_1) = \Psi \bar{v}(\dot{g}_1) = \overline{\Psi_X \circ v(g_1)} = \dot{g}$.

Avec ce dernier résultat on a prouvé que φ est aussi application surjective et donc la proposition est complètement prouvée.

Reçue le 5 novembre 1970

Institut Polytechnique, Jassy,
Chaire de Mathématiques I

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CATEGORII SCINDATE ASOCIATE PREFASCICOLELOR DE MULȚIMI

(Rezumat)

Se demonstrează că dat un prefascicol de mulțimi $P: E^o \rightarrow \text{Ens}$ categoria scindată asociată acestui prefascicol coincide cu categoria E/P .

Se dă după aceea, prin propoziția 1, construcția categoriei E' scindate asociată prefascicolului $P \circ u$ unde $P: E^o \rightarrow \text{Ens}$ iar $u: E' \rightarrow E$ este un functor covariant deplin fidel.

În sfârșit, se demonstrează propoziția 2 care afirmă că la orice pereche de functori

$E \xrightleftharpoons[u]{u} E'$, v adjunct la stînga al lui u , se asociază o altă pereche de adjuncție

$E/P \xrightleftharpoons[v]{\bar{u}} E'/u^*(P)$, unde $P: E^o \rightarrow \text{Ens}$ iar u^* este functorul imagine inversă de prefascicole asociate lui u .

TOPOLOGIES AND ORDERS SIMULTANEOUSLY COMPATIBLE WITH THE OPERATION OF A GROUP AND WHICH INDUCE ONE ANOTHER ¹⁾

BY

GH. COSTOVICI

1. In this Note we are studying the connection between the set of the preorders and orders compatible with the operation of a group and the discrete topologies compatible with the operation of the same group.

According to [1, p. 54], we call discrete topology defined on a set, that topology for which the closure has the property: $\bigcup_{\gamma \in \Gamma} \bar{X}_\gamma = \bigcap_{\gamma \in \Gamma} X_\gamma$, Γ arbitrary.

If the triple (A, B, f) denotes the mapping for which A is the domain, B is the range and f is the correspondance law, then, exercises 17 and 18 given in [1, p. 83] without being solved, are expressed together in this way: the mapping (R_2, T_2, f) where R_2 is the set of the orders (reflexive, antisymmetric and transitive binary relations) defined on a set M , T_2 is the set of the discrete topologies defined on the set M and wich transform M in T_0 -space (a topological space is a T_0 -space $\Leftrightarrow$ distinct points do not have identical closures), and $f(\rho) = \tau$ where the closure in the topology τ is defined by $\bar{A} = \{x; x \in M, \exists a \in A \text{ and } x \rho a\}$ is a bijection.

For the sake of the completeness, we recall that a binary relation ρ defined on a multiplicative and non-commutative group G , is said to be left compatible with the operation of the group $\Leftrightarrow x \rho y \Rightarrow tx \rho ty \forall t \in G$, right compatible $\Leftrightarrow x \rho y \Rightarrow xg \rho yg \forall g \in G$, compatible $\Leftrightarrow x \rho y \Rightarrow axb \rho ayb \forall a, b \in G$.

Similarly, a topology τ defined on a multiplicative and non-commutative group, G , is said to be left compatible with the operation of the

1) Presented at the scientific session of the Polytechnique Institute of Jassy, July 2—4, 1970.

group $\Leftrightarrow \bar{A}x \subset \bar{A}x \quad \forall A \subset G$ and $x \in G$, right compatible $\Leftrightarrow y\bar{A} \subset y\bar{A} \quad \forall A \subset G$ and $y \in G$, and compatible $\Leftrightarrow \bar{AB} \subset \bar{AB} \quad \forall A, B \subset G$.

We note that a discrete topology τ is left and right compatible with the operation of multiplicative and non-commutative group, G , $\Leftrightarrow \tau$ is compatible. Indeed, if the left member is satisfied, then from $b\bar{A} \subset b\bar{A}$ and $\bar{A}b \subset \bar{A}b$, $\forall b \in B$, follows $\bigcup_{b \in B} b\bar{A} = B\bar{A} \subset \bigcup_{b \in B} \bar{A}b = \bar{A}B$ and $\bigcup_{b \in B} \bar{A}b = \bar{A}B \subset \bigcup_{b \in B} b\bar{A} = B\bar{A}$ and therefore $\bar{AB} \subset \bar{AB} \subset \bar{AB} = \bar{AB}$, $\forall A, B \subset G$.

Conversely, if the right member is satisfied, then, $\bar{A}x \subset \bar{A}x \subset \bar{A}x$ and $y\bar{A} \subset y\bar{A} \subset y\bar{A} \quad \forall A \subset G$ and $y \in G$.

We recall that the set of the binary relations defined on a set can be ordered by $\rho_\alpha \leq \rho_\beta \Leftrightarrow x\rho_\alpha y \Rightarrow x\rho_\beta y$ and the set of the topologies defined on a set can be ordered by

$$\tau_\beta \leq \tau_\alpha \Leftrightarrow \bar{A} \subset \bar{A}^{\tau_\alpha}$$

(the closure of A in the topology τ_α is included in the closure of A in the topology τ_β).

2. With the following notations; R_1 —set of the preorders (reflexive and transitive binary relations) defined on a set M ; T_1 —set of the discrete topologies defined on M ; R_2 —set of the orders defined on M ; T_2 —set of the discrete topologies defined on M , and which transform M in T_0 -space; R_3 —set of the left compatible preorders with the operation of a multiplicative and non-commutative group, G ; T_3 —set of the discrete topologies right compatible with the operation of, G ; R_4 —set of the right compatible preorders with the operation of G ; T_4 —set of the discrete topologies, left compatible with the operation of G ; R_5 —set of the compatible preorders with the operation of G ; T_5 —set of the discrete topologies, compatible with the operation of G ; R_6 —set of the left compatible orders with the operation of G ; T_6 —set of the discrete topologies, right compatible with the operation of G , and which transform G in T_0 -space; R_7 —set of the right compatible orders with the operation of G ; T_7 —set of the discrete topologies, left compatible with the operation of G , and which transform G in T_0 -space; R_8 —set of the compatible orders with the operation of G ; T_8 —set of the discrete topologies, compatible with the operation of G , and which transform G in T_0 -space, we have the

Theorem. The mapping (R_i, T_i, f) where f is defined above and $i = 1, 2, \dots, 8$ is a anti-isomorphism of order.

We shall make the proof only for (R_8, T_8, f) , the remaining of the demonstration following in the same way.

So, let $f(\rho) = \tau$ be the topology whose closure is defined by $\bar{A} = \{x; x \in G, \exists a \in A \text{ and } x\rho a\}$, $\rho \in R_8$.

I. $a \rho a \forall a \in A \Rightarrow A \subset \bar{A}$.

II. $x \in \overline{A \cup B} \Leftrightarrow \exists t \in A \cup B$ and $x \rho t \Leftrightarrow (\exists t \in A$ and $x \rho t)$ or $(\exists t \in B$ and $x \rho t) \Rightarrow x \in \bar{A} \cup \bar{B}$. $x \in \bar{A} \cup \bar{B} \Leftrightarrow x \in \bar{A}$ or $x \in \bar{B} \Leftrightarrow (\exists t_1 \in A$ and $x \rho t_1)$ or $(\exists t_2 \in B$ and $x \rho t_2) \Rightarrow (\exists t_1 \in A \cup B$ and $x \rho t_1)$ or $(\exists t_2 \in A \cup B$ and $x \rho t_2) \Rightarrow x \in \overline{A \cup B}$.

III. $x \in \bar{A} \Leftrightarrow \exists t \in \bar{A}$ and $x \rho t \Leftrightarrow \exists t_1 \in A$ and $t \rho t_1$ and $x \rho t \Rightarrow \exists t_1 \in A$ and $x \rho t_1 \Rightarrow x \in \bar{A}$.

IV. $\Phi = \Phi$.

V. $x \in \bigcup_{\gamma \in \Gamma} \bar{X}_\gamma \Leftrightarrow \exists \gamma \in \Gamma$ and $t \in X_\gamma$ and $x \rho t \Leftrightarrow t \in \bigcup_{\gamma \in \Gamma} X_\gamma$ and $x \rho t \Leftrightarrow x \in \bigcup_{\gamma \in \Gamma} X_\gamma$.

VI. $xy \in \bar{A} \bar{B} \Leftrightarrow x \in \bar{A}$ and $y \in \bar{B} \Leftrightarrow \exists a \in A$ and $x \rho a$ and $\exists b \in B$ and $y \rho b \Rightarrow \exists ab \in AB$ and $xy \rho ab \Leftrightarrow xy \in \bar{AB}$.

VII. $\bar{x} = \bar{y} \Rightarrow x \subset \bar{y}$ and $y \subset \bar{x} \Leftrightarrow \exists t_1 \in \{y\}$ and $x \rho t_1$ and $\exists t_2 \in \{x\}$ and $y \rho t_2 \Rightarrow x \rho y$ and $y \rho x \Rightarrow x = y$.

VIII. $\forall \tau \in T_8$, consider ρ' , defined as $x \rho' y \Leftrightarrow x \in \bar{y}$. $x \in \bar{x} \Rightarrow x \rho' x$. $x \rho' y$ and $y \rho' x \Leftrightarrow x \in \bar{y}$ and $y \in \bar{x} \Rightarrow \bar{x} \subset \bar{y} = \bar{y}$ and $\bar{y} \subset \bar{x} = \bar{x} \Rightarrow \bar{x} = \bar{y} \Rightarrow x = y$. $x \rho' y$ and $y \rho' z \Rightarrow x \in \bar{y}$ and $y \in \bar{z} \Rightarrow x \in \bar{y}$ and $\bar{y} \subset \bar{z} = \bar{z} \Rightarrow x \in \bar{z} \Leftrightarrow x \rho' z$. $x \rho' y \Leftrightarrow x \in \bar{y} \Rightarrow axb \in ayb \subset \bar{ayb} \Leftrightarrow axb \rho' ayb$.

IX. $x \rho y \Leftrightarrow x \in \bar{y} \Leftrightarrow x \rho' y$.

X. $\rho_\alpha \leq \rho_\beta \Leftrightarrow x \rho_\alpha y \Rightarrow x \rho_\beta y$. $a \in \bar{A} \Leftrightarrow \exists t \in A$ and $a \rho_\alpha t \Rightarrow \exists t \in A$ and $a \rho_\beta t \Leftrightarrow a \in \bar{A}$ and therefore $\tau_\beta \leq \tau_\alpha$. Conversely, $\tau_\alpha \leq \tau_\beta \Leftrightarrow \bar{A} \subset \bar{A}$. $x \rho_\beta y \Leftrightarrow x \in \bar{y} \Rightarrow x \in \bar{y} \Leftrightarrow x \rho_\alpha y$ and therefore $\rho_\beta \leq \rho_\alpha$.

Received July 28, 1970

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TOPOLOGII ȘI ORDINI SIMULTAN COMPATIBILE CU OPERAȚIA UNUI GRUP ȘI CARE SE INDUC RECIPROC

(Rezumat)

Se studiază legătura dintre preordini, ordini și topologii discrete definite pe o mulțime, precum și compatibilitatea lor cu operația unui grup necomutativ, obținându-se rezultatul: (R_i, T_i, f) , din teoremă, este antiizomorfism de ordine, $i = 1, 2, \dots, 8$.

SUR L'ESTIMATION EN MOYENNE D'ORDRE $2p$ DES FONCTIONS DISCRÈTES ¹⁾

PAR

VIOREL MURGESCU

1. Introduction

Il s'agit de traiter le problème d'approximation des fonctions discrètes (définies sur un ensemble fini) par un élément appartenant à un espace vectoriel à m dimensions, dans l'hypothèse que celui-ci réalise le minimum de l'écart moyen correspondant d'ordre $2p$.

Si le but qu'on cherche à atteindre est d'appliquer la méthode généralisée de Gauss au problème d'estimation des valeurs des fonctions, au fond on traite ici un problème concret sur la meilleure approximation des fonctions discrètes définies dans un R^s , par des éléments d'un espace vectoriel de dimension finie [1].

On établit un théorème d'existence et d'unicité et on indique la possibilité d'appliquer la méthode itérative de Newton—Kantorovitch pour la détermination de l'élément d'estimation.

2. L'énoncé du problème

Soit $X = \{x_k\}$ un ensemble fini inclus dans un intervalle compact K (réel, à s dimensions) et soit encore $f: X \rightarrow F$ une fonction réelle discrète, où $F = \{f_k\}$, $f_k = f(x_k)$, ($k = 1, 2, \dots, n$).

On note encore par $\{\varphi_i(x)\}$, ($i = 1, 2, \dots, m$), un système de fonctions linéairement indépendantes sur K , engendrant l'espace linéaire réel à m dimensions G_m , dont les éléments ont la forme

¹⁾ Présentée lors de la session scientifique de l'Institut Polytechnique de Jassy, le 2—4 juillet 1970.

$$(2.1) \quad g(x) = \sum_{i=1}^m \gamma_i \varphi_i(x), \quad (x \in K, \gamma_i \in R).$$

Définition. On dit que l'élément $g_0(x) \in G_m$ estime en moyenne d'ordre $2p$ la fonction $f(x)$ sur l'ensemble X , si son écart moyen d'ordre $2p$,

$$(2.2) \quad \delta_{2p}(f, g_0) = \left\{ \frac{1}{n} \sum_{k=1}^n [g_0(x_k) - f(x_k)]^{2p} \right\}^{\frac{1}{2p}},$$

satisfait la condition

$$(2.3) \quad \delta_{2p}(f, g_0) = \min_{g \in G_m} \delta_{2p}(f, g), \quad g_0 \in G_m.$$

Le problème qu'on pose dans ce qui suit consiste à établir tout d'abord l'existence d'un élément g_0 afin que la condition (2.3) soit satisfaite. Or, il est bien connu que le problème de l'existence et de l'unicité de g_0 est résolu pour $p=1$ par la méthode des moindres carrés. Donc il reste de l'établir encore pour $p=2,3,\dots$, ce qui équivaut à démontrer qu'on peut déterminer toujours les coefficients γ_i , ($i=1,2,\dots,m$), de (2.1), de façon que $\delta_{2p}(f, g_0)$ soit un minimum.

Dans l'intention, on considère d'abord le cas banal $m \geq n$, pour lequel il existe toujours des solutions; elles vérifient le système linéaire

$$\sum_{i=1}^m \gamma_i \varphi_i(x_k) = f(x_k), \quad (k=1,2,\dots,n), \quad \text{rang}[\varphi_i(x_k)] = m.$$

Donc, il existe en ce cas au moins un élément g_0 égalant f sur X , c'est-à-dire $f \in G_m$ sur X .

Maintenant, si l'on considère dans un R^{m+1} les points

$$(2.4) \quad \Gamma = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_m), \quad \gamma_0 = -1$$

et

$$(2.5) \quad a_k = (f_k, \varphi_{1k}, \varphi_{2k}, \dots, \varphi_{mk}), \quad f_k = f(x_k), \quad \varphi_{ik} = \varphi_i(x_k), \\ (k=1,2,\dots,n; i=1,2,\dots,m),$$

l'écart $\delta_{2p}(f, g)$ peut être écrit sous la forme

$$(2.6) \quad \delta_{2p}(f, g) = \left\{ \frac{1}{n} \sum_{k=1}^n (\Gamma | a_k)^{2p} \right\}^{\frac{1}{2p}},$$

où on a noté

$$(2.7) \quad (\Gamma | a_k) = \sum_{i=1}^m \gamma_i \varphi_{ik} + \gamma_0 f_k, \quad (k=1,2,\dots,n).$$

Par conséquent, le problème qui fut posé auparavant revient au fond, dans le cas $m < n$, au problème d'établir l'existence d'une valeur de l'argument vectoriel Γ de sorte qu'il réalise le minimum de la fonction

$$(2.8) \quad \Phi_{2p}(\Gamma) = \left\{ \sum_{k=1}^n (\Gamma | a_k)^{2p} \right\}^{\frac{1}{2p}}.$$

Ce problème fait le sujet du paragraphe suivant.

3. L'existence et l'unicité de l'élément g_0

Théorème. *Etant donnée une fonction réelle discrète*

$$f: X \rightarrow F, \quad (X = \{x_k\}; \quad F = \{f(x_k)\}; \quad k = 1, 2, \dots, n; \quad E \subset K \subset R^s),$$

où K est un compact à s -dimensions, et G_m étant un espace linéaire à m dimensions engendré par le système de fonctions

$$\{\varphi_i(x)\}, \quad (i = 1, 2, \dots, m; x \in K),$$

alors, quels que soient les entiers positifs m, n et $p, (n > m)$, il existe un élément unique $g_0 \in G_m$ de sorte qu'il estime la fonction f en moyenne d'ordre $2p$ sur $X \subset K$.

Démonstration. Le principe de la démonstration de ce théorème consiste à prouver que la fonction $\Phi_{2p}(\Gamma)$ est strictement convexe et bornée inférieurement sur un ensemble convexe compact. Par conséquent son minimum absolu est atteint sur cet ensemble. Vu encore qu'elle est plusieurs fois différentiable, son point de minimum absolu est un point de minimum relatif et la stricte convexité de la fonction assure sa unicité.

a) On montre premièrement que la fonction $\Phi_{2p}(\Gamma)$ est strictement convexe dans R^m , où $\Gamma = (-1, \gamma_1, \gamma_2, \dots, \gamma_m) \in R^m \subset R^{m+1}$. Pour ce but observons tout d'abord qu'on peut supposer la fonction $\Phi_{2p}(\Gamma)$ strictement positive dans R^m , car le contraire conduirait au cas banal $\min \Phi_{2p}(\Gamma) = 0$, $(\Gamma | a_k) = 0$ pour $k = 1, 2, \dots, n$.

Cette précision faite, il reste à démontrer la stricte convexité de la fonction

$$(3.1) \quad \psi(\lambda) \stackrel{\text{déf}}{=} \Phi_{2p}[\lambda \Gamma_1 + (1 - \lambda) \Gamma_2]$$

sur l'intervalle $[0, 1]$, quels que soient $\Gamma_1, \Gamma_2 \in R^m$, ce qui représente la condition nécessaire et suffisante pour que $\Phi_{2p}(\Gamma)$ soit strictement convexe. Pour ce but on calcule les deux premières dérivées de la fonction

$$(3.2) \quad \psi^{2p}(\lambda) = \sum_{k=1}^n [\lambda (\Gamma_1 | a_k) + (1 - \lambda) (\Gamma_2 | a_k)]^{2p},$$

en obtenant

$$(3.3) \quad \psi'(\lambda) = \psi^{1-2p}(\lambda) \sum_{k=1}^n (\Gamma_1 - \Gamma_2 | a_k) [\lambda (\Gamma_1 | a_k) + (1 - \lambda) (\Gamma_2 | a_k)]^{2p-1}$$

et

$$(3.4) \quad \begin{aligned} \psi''(\lambda) = & \frac{2p-1}{\psi^{4p-1}(\lambda)} \left\{ \sum_{k=1}^n [\lambda (\Gamma_1 | a_k) + (1 - \lambda) (\Gamma_2 | a_k)]^{2p} \times \right. \\ & \times \sum_{k=1}^n (\Gamma_1 - \Gamma_2 | a_k)^2 [\lambda (\Gamma_1 | a_k) + (1 - \lambda) (\Gamma_2 | a_k)]^{2p-2} - \\ & \left. - \left[\sum_{k=1}^n (\Gamma_1 - \Gamma_2 | a_k) [\lambda (\Gamma_1 | a_k) + (1 - \lambda) (\Gamma_2 | a_k)]^{2p-1} \right]^2 \right\}. \end{aligned}$$

En notant

$$(3.5) \quad \begin{aligned} \alpha_k &= (\Gamma_1 - \Gamma_2 | a_k); \quad \beta_k = \lambda (\Gamma_1 | a_k) + (1 - \lambda) (\Gamma_2 | a_k); \\ & (k = 1, 2, \dots, n), \end{aligned}$$

et compte tenu de l'identité de Lagrange, on peut vérifier facilement l'identité

$$(3.6) \quad \begin{aligned} & \left(\sum_{i=1}^n \alpha_i^2 \beta_i^{2p-2} \right) \left(\sum_{j=1}^n \beta_j^{2p} \right) - \left(\sum_{k=1}^n \alpha_k \beta_k^{2p-1} \right)^2 = \\ & = \sum_{\substack{i, j=1 \\ (i < j)}}^n \beta_i^{2p-2} \beta_j^{2p-2} (\alpha_i \beta_j - \alpha_j \beta_i)^2 \end{aligned}$$

qu'on utilise maintenant pour transformer l'expression de $\psi''(\lambda)$, de sorte qu'elle prend la forme

$$(3.4') \quad \psi''(\lambda) = \frac{2p-1}{\psi^{4p-1}(\lambda)} \sum_{\substack{i, j=1 \\ (i < j)}}^n \beta_i^{2p-2} \beta_j^{2p-2} (\alpha_i \beta_j - \alpha_j \beta_i)^2.$$

En remarquant encore qu'on a

$$(3.7) \quad \alpha_i \beta_j - \alpha_j \beta_i = (\Gamma_1 | a_i) (\Gamma_2 | a_j) - (\Gamma_1 | a_j) (\Gamma_2 | a_i), \quad (i, j = 1, 2, \dots, n),$$

on obtient finalement

$$(3.4'') \quad \begin{aligned} \psi''(\lambda) = & \frac{2p-1}{\psi^{4p-1}(\lambda)} \sum_{\substack{i, j=1 \\ (i < j)}}^n [(\Gamma_1 | a_i) (\Gamma_2 | a_j) - (\Gamma_1 | a_j) (\Gamma_2 | a_i)]^2 \times \\ & \times [\lambda (\Gamma_1 | a_i) + (1 - \lambda) (\Gamma_2 | a_i)]^{2p-2} [\lambda (\Gamma_1 | a_j) + (1 - \lambda) (\Gamma_2 | a_j)]^{2p-2}. \end{aligned}$$

Vu que, d'une part, $\psi(\lambda)$ est une fonction strictement positive et d'autre part, les quantités de (3.7) ne peuvent pas être toutes nulles parce

que les vecteurs Γ_1 et Γ_2 de R^{m+1} sont sans cesse linéairement indépendants, il en résulte $\psi''(\lambda) \geq 0$. Mais, $\psi''(\lambda)$ n'est identiquement nulle dans aucun sousintervalle (non trivial) de $[0,1]$ et donc, $\psi(\lambda)$ est strictement convexe dans $[0,1]$.

Par conséquent $\Phi_{2p}(\Gamma)$ étant une fonction strictement convexe, si elle possède un point de minimum, alors ce point est unique.

b) Pour que la démonstration soit complète, il reste à préciser un ensemble compact convexe dans lequel la fonction strictement convexe $\Phi_{2p}(\Gamma)$ soit bornée. Pour ce but on observe premièrement qu'il existe m vecteurs a_k de (2.5) linéairement indépendants (vu que le système $\{\varphi_i(x)\}$, $i = 1, 2, \dots, m$, constitue une base dans G_m) et avec un numérotage convenable de ces vecteurs, on peut les supposer d'être justement $a_1, a_2, \dots, a_m$.

Alors en considérant l'espace réel Ω_m des éléments de la forme $\omega = (-1, \omega_1, \omega_2, \dots)$, il en résulte que la transformation (T) définie par des fonctions linéaires affines

$$(3.8) \quad (T) \quad \omega_k = (a_k | \Gamma), \quad (k = 1, 2, \dots, m),$$

est non singulière et transforme la fonction (2.8) en

$$(3.9) \quad \tilde{\Phi}_{2p}(\omega) = \left[\sum_{i=1}^m \omega^{2p} + \sum_{i=m+1}^n \left(\sum_{j=1}^m \lambda_{ij} \omega_j + \mu_i \right)^{2p} \right]^{\frac{1}{2p}},$$

où λ_{ij} et μ_i sont les nouveaux coefficients des expressions de $(\Gamma | a_k)$ pour $k = m+1, \dots, n$.

Vu que zéro est une barrière inférieure pour $\Phi_{2p}(\omega)$, soit $\omega_0 = (-1, \omega_{10}, \omega_{20}, \dots, \omega_{m0})$ un point arbitrairement choisi dans Ω_m et $\eta_0 = \tilde{\Phi}_{2p}(\omega_0)$ la valeur correspondante de la fonction (3.9). Si η_0 est la borne inférieure de $\tilde{\Phi}_{2p}(\omega)$, alors il n'y a encore rien à démontrer, car $\Gamma_0 = T^{-1}(\omega_0)$ est l'unique point qui réalise le minimum absolu de la fonction $\Phi_{2p}(\Gamma)$.

Donc, en supposant maintenant le contraire, c'est-à-dire qu'il existe des valeurs $\tilde{\Phi}_{2p}(\omega)$ inférieures à η_0 , alors la fonction strictement convexe $\tilde{\Phi}_{2p}(\omega)$ atteint son minimum absolu dans le cube $|\omega_i| \leq \eta_0$, ($i = 1, 2, \dots, m$); par conséquent, son image dans R^m par T^{-1} , qui est aussi un compact convexe, contient un point $\Gamma_0 = (-1, \gamma_{10}, \gamma_{20}, \dots, \gamma_{m0})$, de sorte que $\Phi_{2p}(\Gamma_0) = \inf \Phi_{2p}(\Gamma)$. Vu que $\Phi_{2p}(\Gamma)$ est une fonction continue et plusieurs fois différentiable, Γ_0 est un point de minimum relatif pour cette fonction, dont l'unicité lui fut établie auparavant.

Alors $g_0 = \sum_{i=1}^m \gamma_{i0} \varphi_i(x)$ est l'élément unique de G_m qui estime f en moyenne d'ordre $2p$ sur X .

4. Sur l'estimation de l'élément g_0

On peut déterminer le point $(\gamma_{10}, \gamma_{20}, \dots, \gamma_{m0}) \in R^m$ qui réalise le minimum de la fonction (2.8) comme solution du système algébrique qu'on obtient en annulant toutes les dérivées partielles du premier ordre de la fonction $\Phi_{2p}(\Gamma)$, prises par rapport aux variables $\gamma_1, \gamma_2, \dots, \gamma_m$. Ce système écrit dans une forme matricielle est

$$(4.1) \quad P(\Gamma) = [h_i(\Gamma)] = \left[\frac{1}{2p-1} \sum_{k=1}^n \varphi_{ik}(\Gamma | a_k)^{2p-1} \right] = [0], \quad (i = 1, 2, \dots, m),$$

où Γ et a_k sont les vecteurs (2.4) et (2.5), tandis que l'opérateur P est de manière évidente continu et plusieurs fois différentiable.

Dans le cas particulier qu'on obtient en prenant $p = 1$, (4.1) est un système linéaire compatible, dont l'unique solution sera notée par

$$(4.2) \quad \dot{\Gamma} = (\dot{\gamma}_1, \dot{\gamma}_2, \dots, \dot{\gamma}_m).$$

Puisque pour $p > 1$, (4.1) est un système non linéaire, en général on ne peut pas déterminer sa solution exacte; néanmoins on peut chercher à l'estimer par le procédé itératif de Newton—Kantorovich, en prenant comme élément initial d'estimation la solution (4.2) même, vu qu'il minimise les quantités $(\Gamma | a_k)$ dans le sens qu'il minimise encore l'écart quadratique $\delta_2(f, g)$.

Alors, il s'ensuit que le premier élément d'estimation pour la solution du système (4.1) a l'expression

$$(4.3) \quad \overset{1}{\Gamma} = \dot{\Gamma} - P'(\dot{\Gamma})^{-1} P(\dot{\Gamma}),$$

où

$$(4.4) \quad P'(\dot{\Gamma}) = \left[\frac{\partial h_\alpha}{\partial \gamma_\beta} \right]_{\dot{\Gamma}}, \quad (\alpha, \beta = 1, 2, \dots, m),$$

et

$$(4.5) \quad \dot{\Gamma} = [\dot{\gamma}_1, \dot{\gamma}_2, \dots, \dot{\gamma}_m]^T, \quad \overset{1}{\Gamma} = [\overset{1}{\gamma}_1, \overset{1}{\gamma}_2, \dots, \overset{1}{\gamma}_m]^T,$$

avec la condition $\det P'(\dot{\Gamma}) \neq 0$.

Mais on montrera que cette condition peut être toujours satisfaite. Pour ce but on observe qu'en notant

$$(4.6) \quad \overset{\circ}{\omega}_k = (\dot{\Gamma} | a_k), \quad (k = 1, 2, \dots, m),$$

$$(4.7) \quad \dot{\psi}_\alpha = (\varphi_{\alpha 1} \overset{\circ}{\omega}_1^{p-1}, \varphi_{\alpha 2} \overset{\circ}{\omega}_2^{p-1}, \dots, \varphi_{\alpha n} \overset{\circ}{\omega}_n^{p-1}), \quad (\alpha = 1, 2, \dots, m),$$

on a

$$(4.8) \quad \left(\frac{\partial h_\alpha}{\partial \gamma_\beta} \right)_{\dot{\Gamma}} = \sum_{k=1}^n \varphi_{\alpha k} \varphi_{\beta k} (\dot{\Gamma} | a_k)^{2p-2} = \sum_{k=1}^n \varphi_{\alpha k} \varphi_{\beta k} \overset{\circ}{\omega}_k^{2p-2} = (\dot{\psi}_\alpha | \dot{\psi}_\beta),$$

$$(\alpha, \beta = 1, 2, \dots, m),$$

et par suite, $\det P'(\tilde{\Gamma})$ est un déterminant Gram d'ordre m strictement positif, construit avec les vecteurs (4.7), car on peut considérer ces vecteurs d'être linéairement indépendants dans R^n .

En vérité, si on supposerait qu'il existe m nombres λ_i , dont un au moins est différent de zéro, de manière qu'on ait $\sum_{i=1}^n \lambda_i \dot{\psi}_i = 0$, cette relation impliquerait les relations

$$\omega_j^{p-1} \sum_{i=1}^m \lambda_i \varphi_{ij} = 0, \quad (j = 1, 2, \dots, n).$$

Mais vu qu'on peut supposer qu'il existe au choix m nombres ω_k non nulles ce qui est réalisable par une éventuelle modification dans $\tilde{\Gamma}$, les relations précédentes impliquent

$$\sum_{i=1}^m \lambda_i \varphi_{ij_k} = 0, \quad (k = 1, 2, \dots, m),$$

d'où résulte $\lambda_i = 0$, ($i = 1, 2, \dots, m$), car $\text{rang} [\varphi_{ij_k}] = m$.

Par conséquent, les vecteurs (4.7) étant linéairement indépendants dans R^n , on a

$$(4.9) \quad \Delta = \det [(\dot{\psi}_\alpha | \dot{\psi}_\beta)] > 0, \quad (\alpha, \beta = 1, 2, \dots, m),$$

et donc, la relation (4.3) n'est pas dépourvue de sens.

Le processus itératif peut continuer selon la formule

$$(4.10) \quad \tilde{\Gamma}^n = \tilde{\Gamma}^{n-1} - P'(\tilde{\Gamma}^{n-1})^{-1} P(\tilde{\Gamma}^{n-1}), \quad (n = 1, 2, \dots),$$

et sa convergence peut être assurée, par exemple, par les conditions suffisantes ([2], p. 84—89)

$$a) \quad |h_i(\tilde{\Gamma})| \leq \eta_0, \quad (i = 1, 2, \dots, m),$$

$$b) \quad \max_i \sum_{j=1}^m \left(\frac{\partial h_i}{\partial \gamma'_j} - \frac{\partial h_i}{\partial \gamma''_j} \right) \leq K \max_j |\gamma'_j - \gamma''_j|,$$

$$c) \quad \|P'(\tilde{\Gamma})^{-1}\| \leq \max_i \sum_k \frac{|G_{ik}|}{\Delta} \leq B_0,$$

$$d) \quad B_0^2 \eta_0 K \leq \frac{1}{2},$$

où G_{ik} sont les cofacteurs des éléments $(\dot{\psi}_i | \dot{\psi}_k)$.

Observation. Il est possible que le processus (4.10) soit convergent sans ajouter des conditions supplémentaires à la seule condition de choisir

comme vecteur initial d'estimation celui donné par (4.2); mais il nous manque une démonstration en ce sens, bien que la convergence fut vérifiée dans des divers cas numériques, particuliers.

Reçue le 26 novembre 1970

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ASUPRA APROXIMĂRII FUNCȚIILOR DISCRETE ÎN MEDIE DE ORDINUL $2p$

(Rezumat)

Se aplică metoda lui Gauss generalizată la problema aproximării funcțiilor în medie de ordinul $2p$, tratându-se în fond o problemă concretă de cea mai bună aproximare a funcțiilor discrete. Se stabilește o teoremă de existență și unicitate a elementului aproximant și se indică o metodă iterativă pentru determinarea sa.

ASUPRA FUNCȚIILOR CONVEXE

DE

FULGA STOICA

1. În această Notă se demonstrează o teoremă de prelungire a unei funcționale liniare reale majorată de o funcție convexă și se organizează o anumită clasă de funcții convexe ca spațiu cu scalari pozitivi [2].

2. Fie X un spațiu liniar real. O funcție f definită pe $A \subset X$ cu valori în mulțimea numerelor reale R este convexă pe A dacă oricare ar fi $\lambda \geq 0$ și $\mu \geq 0$ cu $\lambda + \mu = 1$ și oricare ar fi $x, y \in A$ cu $\lambda x + \mu y \in A$ să avem $f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$.

Teoremă. Fie X un spațiu liniar real și C o funcție convexă pe X cu $C(kx) = kC(x)$ pentru orice $x \in X$ și $0 \leq k < 1$ și X_0 un subspațiu liniar al lui X .

Dacă f este o funcțională liniară reală definită pe X_0 încât $f(x) \leq C(x)$ oricare ar fi $x \in X_0$ atunci există o funcțională liniară reală F definită pe X care prelungește pe f și $F(x) \leq C(x)$ oricare ar fi $x \in X$.

Demonstrație. Fie $x_1 \in X$ încât $x_1 \notin X_0$; să notăm cu $X_1 = X_0 \oplus x_1$. Oricare ar fi x', x'' din X_0 avem: $f(x') + f(-x'') = f(x' - x'') \leq C(x' - x'') = C(x' + x_1 - x'' - x_1) \leq C(x' + x_1) + C(-x'' - x_1)$ deci $-C(-x'' - x_1) - f(x'') \leq C(x' + x_1) - f(x')$ și deci există

$$\alpha = \sup_{x \in X_0} \{-C(-x - x_1) - f(x)\} \quad \text{și} \quad \beta = \inf_{x \in X_0} \{C(x + x_1) - f(x)\} \quad \text{și} \quad \alpha \leq \beta.$$

Fie $\alpha \leq \lambda_1 \leq \beta$; avem

$$(*) \quad -C(-x - x_1) - f(x) \leq \lambda_1 \leq C(x + x_1) - f(x), \quad x \in X_0.$$

Orice $x \in X_1$ se scrie unic sub forma $x = y + \lambda x$ cu $y \in X_0$ și $\lambda \in R$. Definim atunci aplicația $F_1: X_1 \rightarrow R$ astfel: $F_1(x) = f(y) + \lambda \lambda_1$. Evident aplicația F_1 este liniară și pentru $x \in X_0$ avem $F_1(x) = f(x)$.

1) Prezentată la sesiunea științifică a Institutului politehnic din Iași, din 2—4 iulie 1970.

Să arătăm acum că pentru orice $x \in X_1$, $F_1(x) \in C(x)$. Dacă $\lambda = 0$, $F_1(x) = f(y) \leq C(y) = C(x)$, prin ipoteză. Să presupunem acum $\lambda > 0$; înlocuind în relația (*) pe x cu $(1/\lambda)y$ avem

$$\lambda_1 \leq C\left(\frac{1}{\lambda}y + x_1\right) - f\left(\frac{1}{\lambda}y\right) = C\left(\frac{1}{\lambda}(y + \lambda x)\right) - \frac{1}{\lambda}f(y) = C\left(\frac{1}{\lambda}x\right) - \frac{1}{\lambda}f(y),$$

de unde $\lambda\lambda_1 \leq \lambda C\left(\frac{1}{\lambda}x\right) - f(y)$ și deci

$$F_1(x) \leq \lambda C\left(\frac{1}{\lambda}x\right) = \begin{cases} C(x) & \text{dacă } 0 < \lambda < 1, \\ \frac{C((1/\lambda)x)}{1/\lambda} \leq C(x) & \text{dacă } \lambda \leq 1, \end{cases}$$

ceea ce demonstrează inegalitatea în cazul $\lambda > 0$. Dacă $\lambda < 0$ din inegalitatea (*) în mod analog obținem:

$$F_1(x) = \lambda\lambda_1 + f(y) \leq -\lambda C\left(-\frac{1}{\lambda}x\right) \begin{cases} = C(x) & \text{dacă } -1 < \lambda < 0, \\ \leq C(x) & \text{dacă } \lambda \leq -1; \end{cases}$$

ultima inegalitate rezultă din „monotonia” funcțiilor convexe. Fie $\mathcal{F}$ mulțimea tuturor prelungirilor liniare ale funcționalei f care sînt majorate de funcția convexă C . În baza primei părți a demonstrației avem $\mathcal{F} \neq \emptyset$. Vom defini în $\mathcal{F}$ următoarea relație de ordine: $F' \leq F''$ dacă și numai dacă F'' este o prelungire a lui F' . Să arătăm că mulțimea $\mathcal{F}$ satisface condiția teoremei lui Zorn. Fie pentru aceasta $\mathcal{F}_0 = \{F_\alpha\}_{\alpha \in I}$ o submulțime total ordonată a lui $\mathcal{F}$. Funcționala F_α , $\alpha \in I$ este definită pe subspațiul liniar $X_\alpha \subset X$. Să notăm cu F° funcționala definită pe $X^\circ = \bigcup_{\alpha \in I} X_\alpha$ prin condiția $F^\circ(x) = F_\alpha(x)$ dacă $x \in X_\alpha$. Funcționala F° este marginea superioară a mulțimii $\mathcal{F}_0$, cum se verifică imediat. În baza teoremei lui Zorn mulțimea $\mathcal{F}$ admite un element maximal, F . Funcționala F este definită pe tot spațiul X . În adevăr, în caz contrar ar exista o prelungire a lui F (conform primei părți a demonstrației) și aceasta ar fi o prelungire a elementului maximal F , ceea ce este absurd. Teorema este astfel complet demonstrată.

Am utilizat în cadrul teoremei de mai sus funcții convexe cu proprietatea că $C(kx) = kC(x)$ pentru $0 \leq k < 1$. Pentru asemenea funcții vom demonstra următoarea

Propoziție. Dacă C este o funcție convexă pe X cu $C(0) = 0$ și $C(\lambda x) = \lambda C(x)$ pentru $0 \leq \lambda < 1$ atunci pentru orice număr real $\mu > 0$, mulțimea $A = \{x/C(x) \leq \mu\}$ este o mulțime convexă și frontiera mulțimii A este $A_0 = \{x/C(x) = \mu\}$.

Demonstrație. Oricare ar fi $x, y \in A$ și $0 \leq \lambda \leq 1$ avem:

$$C(\lambda x + (1 - \lambda)y) \leq \lambda C(x) + (1 - \lambda)C(y) \leq \lambda\mu + (1 - \lambda)\mu = \mu$$

și deci $\lambda x + (1 - \lambda)y \in A$, de unde rezultă că A este convexă. Să demonstrăm acum partea a doua a propoziției. Fie $x_0 \in A$ încît $C(x_0) = \mu$ și $0 < \alpha < 1, \beta > 1, x_1 = \alpha x_0, x_2 = \beta x_0$. Avem $C(x_1) = C(\alpha x_0) = \alpha C(x_0) \leq \leq C(x_0) = \mu$ de unde rezultă că $x_1 \in A$. Atunci pentru $0 \leq \lambda \leq 1$ avem

$$C(\lambda x_0 + (1 - \lambda)x_1) = C((\lambda + \alpha - \lambda\alpha)x_0) \begin{cases} = (\lambda + \alpha - \lambda\alpha)C(x_0) & \text{dacă} \\ 0 \leq \lambda + \alpha - \lambda\alpha < 1 \\ \leq (\lambda + \alpha - \lambda\alpha)C(x_0) & \text{dacă} \\ (\lambda + \alpha - \lambda\alpha) \geq 1 \end{cases} < \mu,$$

deci $(x_0, x_1) \subset A$. Pe de altă parte

$$C(\lambda x_0 + (1 - \lambda)x_2) = C((\lambda + \beta - \lambda\beta)x_0) \geq (\lambda + \beta - \lambda\beta)C(x_0) = = (\lambda + \beta - \lambda\beta)\mu > \mu,$$

deoarece $\lambda + \beta - \lambda\beta > 1$. Rezultă că segmentul $(x_0, x_2) \subset CA$ deci $x_0 \in \text{Fr } A = A_0$.

Reciproc: fie x_0 un punct frontieră a lui A și fie $(x_0, x_1) \subset A$ și $(x_0, x_2) \subset CA$. Dacă $0 \leq \lambda \leq 1$ avem $C(\lambda x_0 + (1 - \lambda)x_1) \leq \mu$ și $C(\lambda x_0 + (1 - \lambda)x_2) > \mu$. Folosind acestea rezultă:

$$\lambda C(x_0) = C(\lambda x_0) = C(\lambda x_0 + (1 - \lambda)x_1 - (1 - \lambda)x_1) \leq C(\lambda x_0 + (1 - \lambda)x_1) + + (1 - \lambda)C(-x_1) \leq \mu + (1 - \lambda)C(-x_1)$$

și

$$\mu < C(\lambda x_0 + (1 - \lambda)x_2) \leq \lambda C(x_0) + (1 - \lambda)C(x_2).$$

Deci au loc inegalitățile

$$\mu < \lambda C(x_0) + (1 - \lambda)C(x_2) \quad \text{și} \quad \mu \geq \lambda C(x_0) - (1 - \lambda)C(x_1).$$

Trecînd la limită pentru $\lambda \rightarrow 1$ avem $C(x_0) \leq \mu$ și $C(x_0) \geq \mu$, de unde $C(x_0) = \mu$ adică $x_0 \in A_0$, ceea ce încheie demonstrația propoziției.

3. În această parte vom demonstra următoarea

Teoremă. *Mulțimea funcțiilor convexe, diferențiabile pe intervalul $[a, b]$, formează un spațiu cu scalari pozitivi [2].*

Demonstrație. Dacă C_1 și C_2 sînt două funcții convexe, diferențiabile pe $[a, b]$ atunci suma lor $C_1 + C_2$ este funcție convexă, diferențiabilă pe $[a, b]$. Oricare ar fi numărul $\lambda \geq 0$ și C o funcție convexă, diferențiabilă pe $[a, b]$ funcția λC este de același tip.

Asociativitatea sumei permite definirea unei infinități de operații. Pentru a dovedi teorema trebuie să mai verificăm proprietatea lui Riesz [2]. Pentru aceasta este suficient să arătăm că dacă

$$(**) \quad C_1(x) + C_2(x) = C_3(x) + C_4(x), \quad \forall x \in [a, b],$$

atunci există y_{ij} , ($i, j = 1, 2$) funcții convexe și diferențiabile pe $[a, b]$ încît $C_1(x) = y_{11}(x) + y_{12}(x)$, $C_2(x) = y_{21}(x) + y_{22}(x)$, $C_3(x) = y_{11}(x) + y_{21}(x)$ și $C_4(x) = y_{12}(x) + y_{22}(x)$; cazul general rezultă prin inducție. Să considerăm deci cazul de mai sus. Derivînd relația (***) obținem $C'_1(x) + C'_2(x) = C'_3(x) + C'_4(x)$ unde conform cu [1], C'_1, C'_2, C'_3, C'_4 sînt funcții monoton crescătoare. Funcțiile monoton crescătoare pe $[a, b]$ formează un spațiu cu scalari pozitivi [2], și deci este satisfăcută proprietatea lui Riesz, adică există $z_{11}, z_{12}, z_{21}, z_{22}$ funcții monoton crescătoare pe $[a, b]$ încît au loc relațiile:

$$C'_1 = z_{11} + z_{12}, \quad C'_2 = z_{21} + z_{22}, \quad C'_3 = z_{11} + z_{21}, \quad C'_4 = z_{12} + z_{22}.$$

Primitivele funcțiilor $z_{i,j}$, ($i, j = 1, 2$) sînt, conform [1], funcții convexe și diferențiabile pe $[a, b]$. Deci ele pot fi luate drept funcțiile y_{ij} care se cereau determinate.

Primită la 23 iulie 1970

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SUR LES FONCTIONS CONVEXES

(Résumé)

On démontre un théorème de prolongement d'une fonction linéaire à valeurs réelles qui est majorée par une fonction convexe.

On définit aussi une structure d'espace à scalaires ≥ 0 pour une certaine classe de fonctions convexes.

ON RATE OF CONVERGENCE OF EXPONENTIAL FORMULAE

BY

Z. DITZIAN

In several papers L. C. Hsu [5], Butzer and Berens [1] and the author [2] and [3] investigated the rate of convergence of the two Exponential formulae for strongly continuous semi-groups of operators. In the above the rate of convergence is related to $w_L(\delta, T(\cdot)f)$, the rectified modulus of continuity of $T(t)f$, defined by

$$(1) \quad w_L(\delta, T(\cdot)f) = \sup \{ \|T(t)f - T(s)f\|; |t - s| < \delta, 0 \leq t, s \leq L \}.$$

In this paper we shall be interested in the relation between the rate of convergence of the Exponential formulae and the rectified modulus of continuity of $\frac{d}{dt}T(t)f$, when the latter is continuous. For the formulae treated we shall also show that relations with higher derivatives of $T(t)f$ would not yield an essentially better result. Let us define $w_{1,L}(\delta, T(\cdot)f)$, for continuous $\frac{d}{dt}T(t)f$, by

$$(2) \quad w_{1,L}(\delta, T(\cdot)f) = \sup \left\{ \left\| \frac{d}{dt}T(t_1)f - \frac{d}{dt}T(t_2)f \right\|; |t_1 - t_2| < \delta, 0 \leq t_1, t_2 \leq L \right\}.$$

Our first result will concern Hille's first exponential formula, and can be stated as follows:

Theorem 1. *Let $T(t)$ be a strongly continuous semi-group of operators from a Banach space into itself, then for continuous $\frac{d}{dt}T(t)f$ and $\tau, t \leq A < L$ we have*

$$(3) \quad \|\exp(tA_\tau)f - T(t)f\| \leq (t^{1/2} + t)\tau^{1/2}w_{1,L}(\tau^{1/2}, T(\cdot)f) + B \cdot \tau,$$

where $A_\tau = \frac{T(\tau) - I}{\tau}$ and B is independent of τ .

Proof. We write first

$$(4) \quad \exp(tA_\tau)f - T(t)f = e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} (T(k\tau)f - T(t)f).$$

Using continuity of $\frac{d}{dt} T(t)f$, we have

$$\begin{aligned} T(k\tau)f - T(t)f &= (k\tau - t) \int_0^1 \frac{d}{d\zeta} T(t + \zeta(k\tau - t))f d\zeta = \\ &\Rightarrow (k\tau - t)T(t)f + (k\tau - t) \int_0^1 \left[\frac{d}{d\zeta} T(t + \zeta(k\tau - t))f - \frac{d}{dt} T(t)f \right] d\zeta. \end{aligned}$$

Substituting the above in (4) and recalling the definition of $w_{1,L}(\delta, T(\cdot)f)$, we obtain for $t \leq A$

$$\begin{aligned} \|\exp(tA_\tau)f - T(t)f\| &\leq \left\| \frac{d}{dt} T(t) e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{(k\tau - t)}{k!} \right\| + \\ &+ e^{-t/\tau} \sum_{k=0}^{[L/\tau]} \left(\frac{t}{\tau}\right)^k \frac{|k\tau - t|}{k!} \left\{ \int_0^1 w_{1,L}(\zeta |k\tau - t|, T(\cdot)f) d\zeta \right\} + \\ &+ e^{-t/\tau} \sum_{k=[L/\tau]+1}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{|k\tau - t|}{k!} \left\{ \left\| \frac{d}{dt} T(t) \right\| + \left\| \int_0^1 \frac{d}{d\zeta} T(t + \zeta(k\tau - t))f d\zeta \right\| \right\} \equiv \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Obviously $I_1 = 0$. To estimate I_2 we recall that for $0 \leq \zeta \leq 1$

$$w_{1,L}(\zeta |k\tau - t|, T(\cdot)f) \leq w_{1,L}(|k\tau - t|, T(\cdot)f) \leq \left(1 + \frac{|k\tau - t|}{\eta}\right) w_{1,L}(\eta, T(\cdot)f),$$

where $\eta \leq \min(L - A, t)$. Choosing $\eta = \tau^{1/2}$ we have

$$I_2 \leq e^{-t/\tau} \sum_{k=0}^{[L/\tau]} \left(\frac{t}{\tau}\right)^k \frac{|k\tau - t|}{k!} w_{1,L}(\tau^{1/2}, T(\cdot)f) +$$

$$\begin{aligned}
& + \tau^{-1/2} e^{-t/\tau} \sum_{k=0}^{[L/\tau]} \left(\frac{t}{\tau}\right)^k (k\tau - t)^2 w_{1,L}(\tau^{1/2}, T(\cdot)f) \leq \\
& \leq w_{1,L}(\tau^{1/2}, T(\cdot)f) \left\{ e^{-t/\tau} \sum_{k=1}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{|k\tau - t|}{k!} + \tau^{-1} e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{(k\tau - t)^2}{k!} \right\} \leq \\
& \leq w_{1,L}(\tau^{1/2}, T(\cdot)f) (\tau^{1/2} t^{1/2} + \tau^{1/2} t),
\end{aligned}$$

where the last step uses a Lemma in Butzer and Berens' book [1, p. 18].

To estimate I_3 we use the estimate

$$\begin{aligned}
& \left\| \frac{d}{dt} T(t)f \right\| \leq \|T(l)\| \cdot \left\| \frac{d}{dt} T(t-l)f \right\| \quad \text{for } 0 \leq l < t, \\
& \sup_{0 \leq t \leq 1} \left\| \frac{d}{dt} T(t)f \right\| = M, \quad \|T(l)\| \leq M_1^l \quad \text{for some } M_1 \geq 1,
\end{aligned}$$

and therefore $\left\| \frac{d}{dt} T(t)f \right\| \leq M \cdot M_1^t$. Now, we can write

$$I_3 \leq M \sum_{k=[L/\tau]+1}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{|k\tau - t|}{k!} \{M_1^{k\tau} + M_1^t\}.$$

For $t \leq A$ this implies via [2] $I_3 = o(\tau)\tau \rightarrow 0^+$ which completes the proof. Q.E.D.

The second exponential formula to be treated is also the second in the list of Hille and Phillips [4, p. 354], and is related to approximation by Bernstein polynomials.

Theorem 2. Under the conditions of Theorem 1 we have for $0 \leq t \leq 1$

$$(5) \quad \left\| \left\{ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} T\left(\frac{k}{n}\right) \right\} f - T(t)f \right\| \leq \frac{3}{4} n^{-1/2} w_{1,1}(n^{-1/2}, T(\cdot)f).$$

Proof. Following the proof of Theorem 1, we write

$$\begin{aligned}
& \left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} T\left(\frac{k}{n}\right) f - T(t)f \right\| \leq \left\| \frac{d}{dt} T(t) \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left(\frac{k}{n} - t\right) \right\| + \\
& + \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left| \frac{k}{n} - t \right| \left\{ \int_0^1 w_{1,1} \left(\zeta \left| \frac{k}{n} - t \right|, T(\cdot)f \right) d\zeta \right\} \leq \\
& \leq \left\{ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left| \frac{k}{n} - t \right| + n^{1/2} \left(\frac{k}{n} - t\right)^2 \right\} w_{1,1}(n^{-1/2}, T(\cdot)f).
\end{aligned}$$

But since $\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left(\frac{k}{n} - t \right)^2 = \frac{t(1-t)}{n} \leq \frac{1}{4n}$ and $\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left| \frac{k}{n} - t \right| \leq \sqrt{\frac{1}{4n}} = \frac{1}{2} n^{-1/2}$, we complete the proof which is analogous in both method and result to that given by Lorentz [6, p. 21] for approximation by Bernstein polynomials.

To conclude we shall show now that for the exponential formula treated, existence and continuity of higher derivatives would not yield a better result up to a constant.

Let B be the Banach space of continuous functions on $[0, \infty)$ satisfying for some fixed $A > 0$ $\lim_{x \rightarrow \infty} e^{-Ax} f(x) = 0$ with the norm $\|f\| = \sup_{x \geq 0} |e^{-Ax} f(x)|$. Let $T(t)f(x) = f(x+t)$, obviously T is strongly continuous from B into itself. Moreover, $x^2 \in B$ and of course $T(t)x^2 \in C^\infty$ with respect to t . However

$$\begin{aligned} \exp(tAx^2) - T(t)x^2 &= e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau} \right)^k \frac{1}{k!} ((x+k\tau)^2 - (x+t)^2) = \\ &= e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau} \right)^k \frac{1}{k!} ((k\tau)^2 - t^2) = t\tau, \end{aligned}$$

since $\sum_{k=1}^{\infty} u^{k-2} k^2 \frac{1}{k!} = u^{-1}(e^u + ue^u)$. This result yields the estimate $\sup_{x \geq 0} |e^{-Ax} t\tau| = t\tau$ which is better than estimate (3) applied to $f = x^2$ by a multiplicative constant only. The same example would imply the analogous result on the exponential formula in Theorem 2.

Received July 17, 1970

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GRADUL DE CONVERGENȚĂ AL FORMULELOR EXPONENȚIALE

(Rezumat)

Se determină gradul de convergență al primei și al celei de a doua formule exponențiale de convergență a lui Hille pentru semigrupuri continue în sens forte.

SOME LINEAR TRANSFORMATIONS OF LAMÉ FUNCTIONS AND ELLIPSOIDAL WAVE FUNCTIONS

BY

F. M. ARSCOTT and G. D. L. REID

1. Notation

Jacobian elliptic functions are employed in the paper, with the usual notation, namely $\operatorname{sn}(z, k)$, $\operatorname{cn}(z, k)$, $\operatorname{dn}(z, k)$, etc. ([3], Chap. XIII).

Considerable use will be made of transformations of the modulus k , so we shall adopt the convention that k shall be a fixed real number in the range $(0, 1)$, and when this modulus is employed we shall not include it in our symbolism — i.e. we shall write $\operatorname{sn} z$ instead of $\operatorname{sn}(z, k)$, etc. Whenever any other modulus than k is employed, we shall always include it.

Lamé's equation is taken in the form

$$(1.1) \quad \frac{d^2 w}{dz^2} + \{h - v(v+1)k^2 \operatorname{sn}^2 z\} w = 0,$$

where without loss of generality we may take $v \geq -1/2$, and the ellipsoidal wave equation in the form

$$(1.2) \quad \frac{d^2 w}{dz^2} - (a + bk^2 \operatorname{sn}^2 z + qk^4 \operatorname{sn}^4 z) w = 0;$$

([4, Chapters XV, XVI] and [1, Chapters IX, X]). It is worth commenting that in most of the applications of the above equations to problems in mathematical physics or engineering, the parameters v in (1.1) and q in (1.2) are inherent in the problem, while the others, h , a and b are in the nature of separation constants or eigenvalue parameters, which are disposable and may be chosen so that solutions exist having certain desirable properties.

The most useful solutions of (1.1) and (1.2) are, respectively, the *Lamé polynomials* and the *ellipsoidal wave functions*. The notation for these functions used here is that of [1], which may be briefly described as follows:

Ellipsoidal wave functions are the only solutions of (1.2) which are doubly-periodic, with periods $2K$ or $4K$ or $2iK'$, $4iK'$, and are of the form

$$(1.3) \quad \operatorname{sn}^\rho z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z F(\operatorname{sn}^2 z),$$

where ρ , σ , $\tau = 0$ or 1 and $F(\operatorname{sn}^2 z)$ is an integral function of $\operatorname{sn}^2 z$. They thus fall into eight types, according to the values taken by ρ , σ , τ . They are also characterized by having the

three alternative properties of: (i) being even or odd in z ; (ii) having $2K$ as a period or an anti-period, i.e. $w(z + 2K) = w(z)$ or $w(z + 2K) = -w(z)$; the latter property is conveniently described by saying that $w(z)$ „has period $4K'$ “; (iii) having $2iK'$ as period or anti-period. These properties are entirely determined by the occurrence or otherwise of the extraneous factors $\operatorname{sn}z$, $\operatorname{cn}z$, $\operatorname{dn}z$, which depends, naturally, on the values taken by ρ , σ , τ . We use the general symbol $\operatorname{el}(z)$ to denote any solution of (1.2) of the form (1.3), and then prefix to this the letters u (for unity), s , c , d to indicate the factors $\operatorname{sn}z$, $\operatorname{cn}z$, $\operatorname{dn}z$ if they occur. Thus any function with $\rho = \sigma = \tau = 0$ is written $\operatorname{uel}(z)$, and a function with $\rho = \tau = 1$, $\sigma = 0$ would be written $\operatorname{sdel}(z)$. Within each of the eight types, there is an infinity of solutions of the form (1.3), since each depends upon two integral parameters n and m , with n ranging over $0, 1, 2, \dots$ and m over $0, 1, 2, \dots, n$. It is convenient to write these eight types of functions in the form

$$(1.4) \quad \begin{aligned} &\operatorname{uel}_{2n}^m(z), \operatorname{sel}_{2n+1}^m(z), \operatorname{cel}_{2n+1}^m(z), \operatorname{del}_{2n+1}^m(z), \operatorname{scl}_{2n+2}^m(z), \\ &\operatorname{sdel}_{2n+2}^m(z), \operatorname{cdel}_{2n+2}^m(z), \operatorname{scdel}_{2n+3}^m(z), \end{aligned}$$

the advantage of the above notation being that every function has precisely m zeros on the real interval $(0, K)$ and $n - m$ zeros on the interval $(K, K + iK')$.

In general, (1.2) possesses no solution of the form (1.3); when q is given such solutions exist only when the parameters a , b take particular values, these pairs of values being in one-to-one correspondence with the functions (1.4), i.e. this is a two-parameter eigenvalue problem, and each eigenvalue pair is simple. We denote the values taken by a , b as $a_{2n}^{(u)m}$, $b_{2n}^{(u)m}$, $a_{2n+1}^{(s)m}$, $b_{2n+1}^{(s)m}$, $a_{2n+1}^{(cd)m}$, $b_{2n+1}^{(cd)m}$, using an obvious notation.

Finally, we must note that the el -functions and the corresponding eigenvalues a , b are naturally functions of the parameter q and of the modulus k ; when necessary we indicate this by writing, in full,

$$(1.5) \quad \operatorname{uel}_{2n}^m(z, q, k^2), \quad a_{2n}^{(u)m}(q, k^2), \quad \text{etc.}$$

The case of Lamé's equation is similar, but simpler. It is known [1, § 9.1] that the only doubly-periodic solutions (with periods $2K$ or $4K$, $2iK'$ or $4iK'$) occur when the parameter ν in (1.1) is an integer, $\nu = n$, and the corresponding solutions are of the form (1.3) but $F(\operatorname{sn}^2x)$ is now a polynomial, in sn^2z of degree $(n - \rho - \sigma - \tau)/2$. These solutions, the so-called *Lamé polynomials*, thus take the form of a polynomial in sn^2z , or such a polynomial multiplied by one or more of the factors $\operatorname{sn}z$, $\operatorname{cn}z$, $\operatorname{dn}z$, the total degree of the function being n . The general symbol E is used to denote any such polynomial, and we prefix the letters u , s , c , d to this as for ellipsoidal wave functions. It is also known that these Lamé polynomials are in one-to-one correspondence with the set of all ellipsoidal wave functions, and the latter reduce to Lamé polynomials as $q \rightarrow 0$; thus

$$(1.6) \quad \left\{ \begin{aligned} \operatorname{uel}_{2n}^m(z, 0, k^2) &= uE_{2n}^m(z, k^2), \quad b_{2n}^{(u)m}(0, k^2) = 2n(2n + 1), \\ a_{2n}^{(u)m}(0, k^2) &= -h_{2n}^{(u)m}(k^2). \end{aligned} \right.$$

For both Lamé polynomials and ellipsoidal wave functions, a normalisation constant remains unspecified in the above definitions; such a constant can be chosen in various ways, (one method, for instance, is to stipulate that the leading term in $F(\operatorname{sn}^2z)$ shall be 1) but the results of this paper are qualitatively independent of the constant, so we shall not specify it particularly.

2. Transformation to Complementary Modulus $k' = \sqrt{1 - k^2}$

Let a new complex variable ζ be introduced by

$$(2.1) \quad z = K + iK' + i\zeta,$$

where K, K' are the usual quarter-periods of the Jacobian elliptic functions, and k' denotes the complementary modulus $\sqrt{1 - k^2}$. Then standard addition formulae for the functions $\text{sn}, \text{cn}, \text{dn}$, combined with Jacobi's imaginary transformation, give [1 § 9.4.2]

$$(2.2) \quad \begin{cases} \text{sn } z = k^{-1} \text{dn } (\zeta, k'), \\ \text{cn } z = -i k' k^{-1} \text{cn } (\zeta, k'), \\ \text{dn } z = -k' \text{sn } (\zeta, k'). \end{cases}$$

Lamé's equation (1.1), when transformed to the independent variable ζ , then becomes

$$(2.3) \quad \frac{d^2 w}{d\zeta^2} + \{h' - v(v+1)k'^2 \text{sn}^2(\zeta, k')\} w = 0, \quad h' = v(v+1) - h,$$

and the ellipsoidal wave equation (1.2) becomes

$$(2.4) \quad \begin{cases} \frac{d^2 w}{d\zeta^2} - \{a' + b'k'^2 \text{sn}^2(\zeta, k') - qk'^4 \text{sn}^4(\zeta, k')\} w = 0, \\ a' = -a - b - q, \quad b' = b + 2q. \end{cases}$$

The fact that (2.3a) is of the same form as the original equation (1.1) leads to a number of useful results, since a solution of (2.3) with variable ζ can often be expressed in terms of solutions of (1.1) with variable z . In eigenvalue problems, the simple relation (2.3b) between the eigenvalue parameters h, h' also leads to useful results. These were first given by Erdélyi in [5], and have been developed by subsequent workers, notably Ince [6], [7], who pointed out the value of (2.3b) in computing eigenvalues of h for Lamé polynomials.

The application of (2.1), (2.2) to the ellipsoidal wave equation was made by the first author of this present paper [2]. Again, the results obtained are of both theoretical and practical value.

It is worth remarking that the formula (2.1) has geometrical significance in two ways:

(i) In the complex plane, one can think of it as moving the origin to the point $K + iK'$ and rotating the axes through a right angle. Brief consideration then shows that the pattern of singularities of the Jacobian elliptic functions is unchanged, but the quarter-periods K, K' are reversed;

for instance, the singularity at $z = iK'$ now becomes the singularity at $\zeta = iK$.

(ii) There are two coordinate systems which made use of elliptic functions, namely:

ellipsoidal coordinates α, β, γ related to Cartesians by

$$(2.5) \quad \begin{cases} x = k^2 l \operatorname{sn} \alpha \operatorname{sn} \beta \operatorname{sn} \gamma, & y = -k^2 l k'^{-1} \operatorname{cn} \alpha \operatorname{cn} \beta \operatorname{cn} \gamma, \\ z = i l k'^{-1} \operatorname{dn} \alpha \operatorname{dn} \beta \operatorname{dn} \gamma \end{cases}$$

and elliptic conal (or sphero-conal) coordinates

$$(2.6) \quad \begin{cases} x = k r \operatorname{sn} \alpha \operatorname{sn} \beta, & y = i k k'^{-1} r \operatorname{cn} \alpha \operatorname{cn} \beta, \\ z = k'^{-1} r \operatorname{dn} \alpha \operatorname{dn} \beta, \end{cases}$$

and these systems are of value, for example, in attacking diffraction problems involving ellipsoids or elliptic cones.

When a transformation of the nature of (2.1) is applied to these coordinate systems (i.e. we write $\alpha = K + iK' + i\alpha^*$, and similarly for β, γ) then the formulae (2.2) show that this is essentially equivalent to interchanging the axes Ox, Oz .

To illustrate the use of the transformation (2.1), consider an ellipsoidal wave function of type $\operatorname{uel}(z, q, k^2)$. By (2.2), this function, when expressed in terms of ζ , has the form $F(k^{-2} \operatorname{dn}^2(\zeta, k'))$, and since $\operatorname{dn}^2(\zeta, k') = 1 - k'^2 \operatorname{sn}^2(\zeta, k')$, this becomes an integral function of $\operatorname{sn}^2(\zeta, k')$. Since this function satisfies (2.4a), it is easy to deduce that it is some multiple of an ellipsoidal wave function $\operatorname{uel}_{2n'}^{m'}(\zeta, -q, k'^2)$, and simple consideration of the number of zeros of this function shows that $n' = n$, $m' = n - m$. So we have

$$(2.7) \quad \operatorname{uel}_{2n}^m(z, q, k^2) = \operatorname{uel}_{2n}^{n-m}(\zeta, -q, k'^2),$$

where it is understood that the two functions are equal to within a constant multiple, depending on the normalization used.

We have also the corresponding relation between eigenvalues,

$$(2.8) \quad \begin{cases} a_{2n}^{(u) n-m}(-q, k'^2) + b_{2n}^{(u) n-m}(-q, k'^2) + a_{2n}^{(u) m}(q, k^2) = q, \\ b_{2n}^{(u) n-m}(-q, k'^2) + b_{2n}^{(u) m}(q, k^2) = 2q. \end{cases}$$

For the other seven types of functions the results are, broadly, similar. Since the transformation (2.1) has the effect of interchanging the functions sn and dn , but taking the function cn over into itself, we find that ellipsoidal wave functions of types $\operatorname{uel}, \operatorname{cel}, \operatorname{sdel}, \operatorname{scdel}$ are transformed into functions of the same type, while those of types $\operatorname{sel}, \operatorname{del}, \operatorname{scel}, \operatorname{cdel}$, go over respectively into types $\operatorname{del}, \operatorname{sel}, \operatorname{cdel}, \operatorname{scel}$.

There are five first-order transformations of Jacobian elliptic functions, which are listed on p. 369 of reference [3], and there denoted by

the letters $A - E$. The above formulae arise, essentially from use of transformation (B). The rest of this paper examines the results of applying the other four transformations; for simplicity we consider only the ellipsoidal wave equation since results for Lamé's equation are immediately obtainable by setting $q = 0$, $b = \nu(\nu + 1)$, $a = -h$.

3. Transformation to Pure Imaginary Modulus

Let us now introduce the variable ζ_1 given by

$$(3.1) \quad z = K + \frac{\zeta_1}{k'},$$

and let

$$(3.2) \quad k_1 = \frac{ik}{k'}.$$

Then transformation (A), of [3, p. 369] gives

$$(3.3) \quad \operatorname{sn} z = \operatorname{cn}(\zeta_1, k_1), \quad \operatorname{cn} z = -\operatorname{sn}(\zeta_1, k_1), \quad \operatorname{dn} z = \operatorname{dn}(\zeta_1, k_1)$$

and under the transformation (3.1), the ellipsoidal wave equation (1.2) becomes

$$(3.4) \quad \frac{d^2 w}{d\zeta_1^2} - \{a_1 + b_1 k_1^2 \operatorname{sn}^2(\zeta_1, k_1) + q_1 k_1^4 \operatorname{sn}^4(\zeta_1, k_1)\} w = 0,$$

where

$$(3.5) \quad k'^2 a_1 = a + b k^2 + q k^4, \quad b_1 = b + 2q k^2, \quad q_1 = k'^2 q.$$

Now consider the function $\operatorname{uel}_{2n}^m(\zeta_1, q_1, k_1^2)$; this is a solution of (3.4). As a function of ζ_1 it is an integral function of $\operatorname{sn}^2(\zeta_1, k_1)$, has periods $2k'K$, $2ik'K'$, and hence also the period $2k'(K + iK')$; it is even, and has m zeros in $\zeta_1 \in (0, k'K)$ and $n - m$ zeros in $\zeta_1 \in (k'K, k'(K + iK'))$. Consequently, when we regard it as a function of z it is an integral function of $\operatorname{cn}^2 z$ (and hence of $\operatorname{sn}^2 z$), it satisfies (1.2), is even, and has periods $2K$, $2iK'$. Moreover, it has m zeros in $z \in (K, 2K)$, which by periodicity implies that it has m zeros in $z \in (0, K)$, and similarly $n - m$ zeros in $z \in (K, K + iK')$. It can therefore be none other than a constant multiple of $\operatorname{uel}_{2n}^m(z, q, k^2)$, i.e.

$$(3.6) \quad \operatorname{uel}_{2n}^m(\zeta_1, q_1, k_1^2) = \operatorname{uel}(z, q, k^2),$$

where this relationship holds in the same sense as (2.7). The corresponding relations between the eigenvalues a , b are

$$(3.7) \quad \begin{cases} k'^2 a_{2n}^{(u)m}(q_1, k_1^2) = a_{2n}^{(u)m}(q, k^2) + k^2 b_{2n}^{(u)m}(q, k^2) + q k^4, \\ b_{2n}^{(u)m}(q_1, k_1^2) = b_{2n}^{(u)m}(q, k^2) + 2q k^2. \end{cases}$$

Similar reasoning can be applied to the other seven types of function; since the effect of the transformation (3.1) is to interchange sn and cn , we find that functions of the types uel , del , scl , scdel go over into functions of the same type, while those of types sel , cel , sdel , cdel go over into cel , sel , cdel , sdel respectively. The transformation formulae are as follows: for simplicity the numerical suffixes are omitted since they are the same in every element occurring in the same equation, and the arguments q , k^2 are also omitted. Formulae (3.8) hold in the sense that constant multipliers are omitted, but (3.9), (3.10) are, of course, exact.

$$(3.8) \quad \begin{cases} \text{sel}(\zeta_1, q_1, k_1^2) = \text{cel } z, & \text{cel}(\zeta_1, q_1, k_1^2) = \text{sel } z, \\ \text{del}(\zeta_1, q_1, k_1^2) = \text{del } z, & \text{scl}(\zeta_1, q_1, k_1^2) = \text{scl } z, \\ \text{sdel}(\zeta_1, q_1, k_1^2) = \text{cdel } z, & \text{cdel}(\zeta_1, q_1, k_1^2) = \text{sdel } z, \\ & \text{sdel}(\zeta_1, q_1, k_1^2) = \text{scdel } z; \end{cases}$$

$$(3.9a) \quad k'^2 a^{(d)}(q_1, k_1^2) = a^{(d)} + k^2 b^{(d)} + qk^4,$$

$$(3.10a) \quad b^{(d)}(q_1, k_1^2) = b^{(d)} + 2qk^2$$

and similarly for (sc), (scd), while

$$(3.9b) \quad k'^2 a^{(s,c)}(q_1, k) = a^{(c,s)} + k^2 b^{(c,s)} + qk^4,$$

$$(3.10b) \quad b^{(s,c)}(q_1, k_1^2) = b^{(c,s)} + 2qk^2,$$

(taking either the first or the second character throughout) and similarly for (sd, cd).

4. Transformation to Real Modulus, greater than 1

The transformation (C) of [3, p. 369] gives a somewhat simpler result than the two described above, since it leaves the function $\text{sn } z$ unchanged and interchanges cn , dn . Let us set

$$(4.1) \quad \zeta_2 = kz, \quad k_2 = \frac{1}{k};$$

then we have

$$(4.2) \quad k \text{sn } z = \text{sn}(\zeta_2, k_2), \quad \text{cn } z = \text{dn}(\zeta_2, k_2), \quad \text{dn } z = \text{cn}(\zeta_2, k_2)$$

so that (1.3) becomes

$$(4.3) \quad \frac{d^2 w}{d\zeta_2^2} - \{a_2 + b_2 k_2^2 \text{sn}^2(\zeta_2, k_2) + q_2 k_2 \text{sn}^4(\zeta_2, k_2)\} w = 0,$$

where $a_2 = a/k^2$, $b_2 = b$, $q_2 = qk^2$. In view of the simple relationship between ζ_2 and z , consideration of the zeros of ellipsoidal wave functions presents no problems, and we easily obtain

$$(4.4) \quad \begin{cases} \text{uel}_{2n}^m(\zeta_2, q_2, k_2^2) = \text{uel}_{2n}^m(z, q, k^2), \text{ i.e.} \\ \text{uel}_{2n}^m(kz, qk^2, k^{-2}) = \text{uel}_{2n}^m(z, q, k^2), \end{cases}$$

and similarly for functions of types sel, cdel, scdel, while

$$(4.5) \quad \text{cel}_{2n+1}^m(kz, qk^2, k^{-2}) = \text{del}_{2n+1}^m(z, q, k^2)$$

and similarly for the pairs scel, sdel.

Corresponding relations for the eigenvalues are easily seen to be of the forms

$$(4.6) \quad k^2 a^{(u)}(qk, k^{-2}) = a^{(u)}(q, k^2), \quad b^{(u)}(qk^2, k^{-2}) = b^{(u)}(q, k^2),$$

(and similarly for types (s), (cd), (scd)) and

$$(4.7) \quad k^2 a^{(c)}(qk^2, k^{-2}) = a^{(d)}(q, k^2), \quad b^{(c)}(qk^2, k^{-2}) = b^{(d)}(q, k^2),$$

(and similarly for the pairs (sc), (sd)). An interesting illustration of these relationships is provided by the explicit expression for Lamé polynomials of degree 3, given on [1, p. 205].

5. Two Further Transformations

The above three transformations each have the effect, it will be noted, of interchanging two of the three functions sn, cn, dn. The remaining two transformations of the list on [3, p. 369], namely (D) and (E) are somewhat more complicated in that each produces a cyclic interchange of sn, cn, dn. Indeed, it would seem that (D) is effectively the transformations (C) and (B) performed successively, in that order, while (E) is (A) followed by (B). We shall therefore only describe very briefly the effect of these transformations on the ellipsoidal wave equation.

If we set

$$(5.1) \quad z = K + iK' - \frac{i}{k'} \zeta_3, \quad k_3 = \frac{1}{k'},$$

then (by (D),

$$(5.2) \quad k \operatorname{sn} z = \operatorname{cn}(\zeta_3, k_3), \quad k \operatorname{cn} z = -i k' \operatorname{dn}(\zeta_3, k_3), \quad \operatorname{dn} z = \operatorname{sn}(\zeta_3, k_3).$$

Consequently, the only types of ellipsoidal wave functions which are unaffected by the substitution (5.1) are those of types uel and scdel; sel, cel and del-type functions of z become respectively cel, del and sel functions of ζ_3 , while scel, sdel, cdel functions become cdel, scel, sdel func-

tions. Because of the factor $-i$ multiplying ζ_3 , the intervals $z \in (0, K)$, $z \in (K, K + iK')$ are effectively interchanged for ζ_3 , so that the numerical suffices n, m become $n, n - m$ as in the transformation described in § 2 above.

Finally, we consider (E). Setting

$$(5.3) \quad z = K - \frac{i}{k} \zeta_4, \quad k_4 = -\frac{ik'}{k},$$

we have

$$(5.4) \quad \operatorname{sn} z = \operatorname{dn}(\zeta_4, k_4), \quad k \operatorname{cn} z = ik' \operatorname{sn}(\zeta_4, k_4), \quad \operatorname{dn} z = k' \operatorname{cn}(\zeta_4, k_4),$$

so that $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$ are again interchanged cyclically, but in the reverse direction to that of (5.2). With this difference, the effects of this transformation are similar to that described above, and the relevant formulae may be written down with little trouble.

Received February 10, 1970

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UNELE TRANSFORMĂRI LINIARE ALE FUNCȚIILOR LAMÉ ȘI FUNCȚIILOR DE UNDĂ ELIPSOIDALE

(Rezumat)

Se utilizează unele rezultate ale teoriei transformării funcțiilor eliptice pentru obținerea de noi transformări ale ecuațiilor lui Lamé și a undelor elipsoidale, care prezintă importanță atât din punct de vedere teoretic cât și practic.

A NOTE ON THE HERMITE POLYNOMIALS

BY

W. A. AL-SALAM

1. N. Nielsen [4] gave the formula

$$(1) \quad H_n(x)H_m(x) = \sum_{k=0}^m 2^k k! \binom{n}{k} \binom{m}{k} H_{n+m-2k}(x),$$

and its inverse

$$(2) \quad H_{n+m}(x) = \sum_{k=0}^m (-1)^k 2^k k! \binom{m}{k} \binom{n}{k} H_{n-k}(x)H_{m-k}(x),$$

where $m \leq n$ and $H_n(x)$ is the n^{th} Hermite polynomial. Burchnell [1] gave an elegant proof of (1) and (2) based on the operational representation

$$(3) \quad H_n(x) = (-1)^n (D - 2x)^n \cdot 1, \quad D = \frac{d}{dx}.$$

We therefore think that it might be of interest to give another proof of (1) and (2) based on the operational representation

$$(4) \quad e^{-D^2} x^n = H_n\left(\frac{x}{2}\right), \quad (n = 0, 1, 2, \dots).$$

We shall also point out that, in some sense, formula (1) characterizes the Hermite polynomials.

2. It is well known that if

$$f(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-y)^2}{4}\right\} \Phi(y) dy,$$

then its inversion (which holds in case $\Phi(x)$ a polynomial) is given by

$$\Phi(x) = e^{-D^2} f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(2k)}(x).$$

We thus conclude from Howell's formula [3]

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-(x-y)^2] H_n(x) H_m(x) dx = m! 2^n y^{n-m} L_m^{(n-m)}(-2y^2), \quad (m \leq n),$$

that

$$(5) \quad H\left(\frac{x}{2}\right) H_m\left(\frac{x}{2}\right) = m! 2^m e^{-D^2} \left\{ x^{n-m} L_m^{(n-m)}\left(-\frac{x^2}{2}\right) \right\},$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial defined by means of

$$(6) \quad L_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}.$$

If we now rewrite formula (6)

$$(7) \quad m! 2^m x^{n-m} L_n^{(n-m)}\left(-\frac{x^2}{2}\right) = m! \sum_{k=0}^m 2^k \binom{m}{k} \frac{x^{m+n-2k}}{(n-k)!}$$

and if we apply the operator e^{-D^2} to both sides of (7) and using (5) and (7) we get formula (1).

To obtain (2) we simply apply e^{-D^2} to the expansion [5, p. 207]

$$x^{n+m} = n! 2^n \sum_{k=0}^m (-1)^k \binom{m}{k} x^{m-k} L_{n-k}^{(m-n)}\left(-\frac{x^2}{2}\right).$$

3. Another application of the operational formula (4) is the simple derivation of some rather elegant formulas that Professor L. Carlitz has recently communicated to this author, namely,

$$(8) \quad \sum_{n,m=0}^{\infty} H_{n+m}(z) H_m(b) H_n(c) \frac{x^m y^n}{m! n!} = \\ = (1 - 4x^2 - 4y^2)^{-1/2} \exp \left\{ \frac{-4a(x^2 + y^2) + 4a(bx + cy) - 4(bx + cy)^2}{1 - 4x^2 - 4y^2} \right\}.$$

The proof we present here depends on the formula [2]

$$e^{-D^2} e^{-kx^2} = (1 - 4k)^{-1/2} \exp \left(\frac{-kx^2}{1 - 4k} \right).$$

We recall that $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}$ from which it follows that

$$\sum_{n,m=0}^{\infty} H_m(b) H_n(c) \frac{x^m y^n}{m! n!} a^{n+m} = e^{2a(bx+cy)-a^2(x^2+y^2)}.$$

Operate now on both sides with e^{-D^2} where $D = d/da$. We get

$$\begin{aligned} \sum_{n,m=0}^{\infty} H_m(b) H_n(c) H_{n+m} \left(\frac{a}{2} \right) \frac{x^m y^n}{m! n!} &= e^{-D^2} e^{2a(bx+cy)-a^2(x^2+y^2)} = \\ &= \exp \left\{ \frac{(bx+cy)^2}{x^2+y^2} \right\} e^{-D^2 - \frac{bx+cy}{x^2+y^2} D} e^{-(x^2+y^2)a^2} = \\ &= \exp \left\{ \frac{(bx+cy)^2}{x^2+y^2} \right\} e^{-\frac{bx+cy}{x^2+y^2} D} (1-4x^2-4y^2)^{-1/2} \exp \left(-\frac{x^2+y^2}{1-4x^2-4y^2} a^2 \right) = \\ &= (1-4x^2-4y^2)^{-1/2} \exp \left[\frac{(bx+cy)^2}{x^2+y^2} - \frac{(x^2+y^2) \left(a - \frac{bx+cy}{x^2+y^2} \right)}{1-4x^2-4y^2} \right]. \end{aligned}$$

After simple calculations we get the required formula (8).

The same technique can be used to prove analogous formulas (also due to Carlitz) to find the sum

$$\sum_{m,n,p=0}^{\infty} H_{n+p}(a) H_{p+m}(b) H_{m+n}(c) \frac{x^m y^n z^p}{m! n! p!}.$$

4. The Hermite polynomials belong to a very large class of polynomial sets called „Appell polynomials“, i.e., polynomial sets $\{F_n(x)\}$ with a generating function

$$(9) \quad A(t) e^{xt} = \sum_{n=0}^{\infty} \frac{F_n(x) t^n}{n!}.$$

It is also known that the Hermite polynomials have the generating relation of the form

$$(10) \quad F_m(x-t) A(t) e^{xt} = \sum_{n=0}^{\infty} \frac{F_{n+m}(x) t^n}{n!}.$$

It is easy to show that a polynomial set $\{F_n(x)\}$ satisfy (9) and (10) if and only if it is the set of Hermite polynomial.

It is also easy to show that (10) holds if and only if

$$(11) \quad F_{n+m}(x) = \sum_{r=0}^{\infty} (-1)^r \binom{m}{r} \binom{n}{r} r! F_{n-r}(x) F_{m-r}(x)$$

so that we have the

Theorem. *An Appell set of polynomials which also satisfies a Nielsen formula (11) must necessarily be the Hermite polynomial set.*

Received April 10, 1970

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NOTĂ ASUPRA POLINOAMELOR HERMITE

(Rezumat)

Utilizând reprezentarea operațională (4) se deduc formulele (1) și (2) referitoare la polinoamele Hermite.

SUR LES CONDITIONS DE SÉPARABILITÉ DES VARIABLES DANS L'ÉQUATION DE LAPLACE EN ESPACE RIEMANNIEN

PAR

J. STEIGENBERGER

1. Introduction

En 1964, M. Ober-Kassebaum [1] a traité le problème de séparation des variables pour les équations de Schroedinger et de Laplace. Il y a donné des conditions suffisantes et nécessaires pour l'existence, en système donné des coordonnées, d'une variété de solutions en forme

$$u(x^1, \dots, x^n) = \frac{1}{P(x^1, \dots, x^n)} \prod_{v=1}^n u_v(x^v),$$

où les fonctions $u_v(x^v)$ dépendent chacune d'une seule variable x^v et, de plus, de certains paramètres. La fonction P , dont l'existence est assurée par les conditions, est un facteur commun à toutes les solutions (P -séparation). Dans la suite, nous ne considérons que le cas $P(x) \equiv 1$ (séparation complète). En ce cas, les conditions de séparabilité se simplifient et elles peuvent être utilisées aisément pour la construction des équations séparables. Au travail cité, on donne aussi des conditions équivalentes de séparabilité sous forme différentielle pour coefficients qui se trouvent dans l'équation différentielle considérée. Au cas de l'équation de Schroedinger, l'utilisation directe de ces conditions est possible pour décider si une équation donnée sera séparable ou non. Mais au cas de l'équation de Laplace, les conditions, contenant encore une fonction inconnue, ne sont guère applicables à cet effet. Il faut les mettre sous une forme plus convenable et c'est le sujet du présent travail.

2. Définitions

Soit $x = (x^1, \dots, x^n)$ un point d'un espace V_n riemannien ou pseudo-riemannien avec un tenseur métrique $g_{ij}(x)$ de sorte que $g_{ij}(x) \in C^2$ et $g \equiv \det(g_{ij}) \neq 0$. Désignons le tenseur métrique contravariant par g^{ij} , le

tenseur de Riemann-Christoffel par R_{ijkl} et le tenseur de Ricci par $R_{jk} \equiv R_{ijk}^i$. La différentiation partielle est indiquée par une virgule : $a_{,\lambda} \equiv \frac{\partial a}{\partial x^\lambda}$, différentiation covariante par $\nabla_\lambda a$. Chaque indice passe de 1 à n ; nous acceptons la convention de sommation pour les indices latins seulement.

Définissant comme opérateur de Laplace l'opérateur $\Delta := g^{ij} \nabla_i \nabla_j$, nous allons considérer l'équation pour une fonction invariante $u(x)$

$$(1) \quad 0 = \Delta u \equiv (g^{rs} u_{,s})_{,r} + \frac{1}{2} (\log |g|)_{,r} g^{rs} u_{,s}.$$

Si les coordonnées sont orthogonales, c'est-à-dire si le tenseur g_{ij} , en ces coordonnées, a la forme diagonale $g_{\kappa\lambda} = \delta_\kappa^\lambda g_\kappa$, $g^{\kappa\lambda} = \delta_\lambda^\kappa g^\kappa$, $g^\kappa = (g_\kappa)^{-1}$, (δ_λ^κ : symbole de Kronecker), l'équation (1) s'écrit

$$(2) \quad 0 = \Delta u \equiv \sum_{v=1}^{\infty} g^v [u_{,vv} + (\log |\sqrt{|g|} g^v)|_{,v} u_{,v}] \equiv \frac{1}{|\sqrt{|g|}} \sum_{v=1}^n [|\sqrt{|g|} g^v u_{,v}]_{,v}.$$

La définition suivante introduit une notion centrale de la théorie de séparation.

Définition 1. Nous dirons que les n fonctions $g^\mu(x) \neq 0$ forment une métrique du type de Staeckel, si existent n^2 fonctions $\psi_\nu^\mu(x^\nu)$, chacune dépendant d'une seule variable, de telle sorte que $\psi \equiv \det(\psi_\nu^\mu) \neq 0$ et

$$(3) \quad \psi_i^\mu g^i = \delta_1^\mu.$$

Naturellement, les fonctions ψ_ν^μ doivent être de la classe C^2 , comme aussi certaines fonctions, qui sont introduites en ce qui suit.

Les conditions suffisantes et nécessaires pour cette propriété (3) d'un système de fonctions g^μ sont données par le lemme suivant. D'abord, nous introduisons les opérateurs différentiels linéaires, symétriques par rapport aux indices κ, λ ,

$$(4) \quad L_{\lambda\kappa}[f] := f_{,\kappa\lambda} - g_\kappa g_{,\lambda}^\kappa f_{, \kappa} - g_\lambda g_{,\kappa}^\lambda f_{, \lambda},$$

qui sont adjoints au système des fonctions g^μ . Puis, nous avons (voir [1] ... [3]) le

Lemme 1. Les n fonctions $g^\mu(x) \neq 0$ forment une métrique staeckelienne, si, et seulement si, elles satisfont aux équations différentielles

$$(5) \quad L_{\kappa\lambda}[g^\mu] = 0, \quad (\kappa \neq \lambda).$$

3. Séparabilité, conditions connues

Nos investigations se rapportent à la notion de séparabilité, qui est donnée, suivant [1], par la

Définition 2. L'équation (1) est nommée (complètement) séparable, s'il existe un système de n équations différentielles ordinaires d'ordre deux de la forme

$$(6) \quad u''_v + F_v(x^v, u_v, u'_v; \alpha_1, \dots, \alpha_{n-1}) = 0$$

contenant $n - 1$ paramètres α de telle manière, que

$$(6') \quad \text{rang} \left(\frac{\partial F_v}{\partial \alpha_\mu} \right) = n -$$

et, $u_v(x^v; \alpha)$ étant des solutions quelconques de (6), $u(x; \alpha) = \prod_{v=1}^n u_v(x^v; \alpha)$ est une solution de l'équation (1).

Voyons, par exemple, le cas trivial de l'équation ondulatoire $u_{xx} - u_{tt} = 0$ (équation de Laplace en espace pseudo-euclidien V_2 , référé à des coordonnées rectilignes x, t). Là, nous avons les équations $u'_1(x) - \alpha u_1(x) = 0$, $u''_2(t) + \alpha u_2(t) = 0$ du type (6), (6'), dont chaque solution u_1, u_2 donne lieu à une solution $u = u_1 u_2$ de l'équation à dérivées partielles.

Généralement, l'existence d'une telle variété des solutions où les variables sont séparées multiplicativement, dépend de certaines propriétés des fonctions g^{ij} , figurant dans l'équation (1). Grâce à [1], nous avons le

Théorème 1. *Pour que l'équation (1) soit séparable, il faut, et il suffit, que les conditions suivantes soient réalisées:*

(A1) *Les coordonnées sont orthogonales, c'est-à-dire $g^{\lambda\lambda} = \delta^\lambda_\lambda g^{\lambda\lambda}$.*

(B1) *Les fonctions $g^{\mu\mu}$ forment une métrique conforme-staekelienne, c'est-à-dire il y a $n^2 + 1$ fonctions $\sigma(x)$, $\psi^\lambda_\lambda(x^\lambda)$ avec $\psi = \det(\psi^\lambda_\lambda) \neq 0$ de sorte que*

$$(7) \quad g^{\mu\mu} = e^{-2\sigma} G^{\mu\mu}, \quad \psi^\mu_\mu G^{\mu\mu} = \delta^{\mu\mu}$$

(autrement dit: il existe une telle fonction $\sigma(x)$ que $G^{\mu\mu} = e^{2\sigma} g^{\mu\mu}$ est du type de Staekel).

(C1) *Il y a de plus n fonctions $h_\mu(x^\mu)$ de sorte que*

$$(8) \quad \psi e^{2\sigma} \prod_{v=1}^n |g^v|^{1/2} = \prod_{\mu=1}^n h_\mu.$$

Tenant compte de (7₁), la condition (8) s'écrit

$$(8') \quad \psi e^{(2-n)\sigma} \prod_{v=1}^n |G^v|^{1/2} = \prod_{\mu=1}^n h_\mu.$$

À l'aide du théorème 1, il est possible de construire des équations séparables du type (1). Étant $n > 2$, nous donnons ψ_ν^μ , h_λ arbitrairement, formons les G^μ suivant (7₂), σ suivant (8') et, enfin, les fonctions g^μ résultent de (7₁). Au cas $n = 2$, nous formons les G^μ suivant (7₂) avec des fonctions ψ_ν^μ arbitraires; l'équation (8'), où ne figurent que les fonctions ψ_ν^μ et h_λ (aussi arbitraires sous la condition $h_1 h_2 \psi > 0$), donne lieu à une restriction du caractère arbitraire des fonctions ψ_ν^μ , la fonction σ reste indéterminée.

Mais, en général, ce théorème ne donne pas la possibilité de décider si une équation (1) avec des fonctions g^μ données est séparable ou non, parce que la vérification directe de l'existence des fonctions σ , ψ_ν^μ , h_λ est, excepté les cas très simples, pratiquement impossible. Nous allons montrer une possibilité qui permet à décider à l'aide de certains calculs simples.

4. Conditions différentielles équivalentes

Dans la suite, soit toujours satisfaite la condition (A1). D'abord, nous ne nous occupons pas que de la condition (B1).

Les g^μ forment une métrique conforme-staeckelienne si, et seulement si, les G^μ sont du type de Staeckel et ainsi, suivant lemme 1, nous avons

$$(9) \quad \bar{L}_{\kappa\lambda}[G^\mu] \equiv G_{,\kappa}^\mu G_{,\lambda}^\mu - G_{,\kappa} G_{,\lambda}^\mu G_{,\kappa}^\mu - G_{,\lambda} G_{,\kappa}^\mu G_{,\lambda}^\mu = 0, \quad (\kappa \neq \lambda).$$

Mettant $\tau = e^{-2\sigma}$, $G^\mu = \frac{1}{\tau} g^\mu$ et tenant compte de $G_{,\kappa}^\mu = \frac{1}{\tau} g_{,\kappa}^\mu - \frac{1}{\tau^2} \tau_{,\kappa} g^\mu$,

$G_{,\kappa\lambda}^\mu = \frac{1}{\tau} g_{,\kappa\lambda}^\mu - \frac{1}{\tau^2} (\tau_{,\kappa} g_{,\lambda}^\mu + \tau_{,\lambda} g_{,\kappa}^\mu + \tau_{,\lambda} g_{,\kappa}^\mu) + \frac{2}{\tau^3} \tau_{,\kappa} \tau_{,\lambda} g^\mu$, nous obtenons

$$(10) \quad \bar{L}_{\kappa\lambda}[G^\mu] \equiv \frac{1}{\tau} L_{\kappa\lambda}[g^\mu] - \frac{1}{\tau^2} g^\mu L_{\kappa\lambda}[\tau].$$

D'ici, nous pouvons remplacer la condition (B1) par la condition:

(B2) *Il existe une telle fonction $\tau(x) \neq 0$ que les équations différentielles*

$$(11) \quad g^\mu L_{\kappa\lambda}[\tau] = \tau L_{\kappa\lambda}[g^\mu], \quad (\kappa \neq \lambda)$$

sont satisfaites.

Pour simplifier cette condition, nous observons que, si une fonction τ satisfait à l'équation (11), il s'en suit de (11) (après division par τg^μ)

$$(12) \quad g_1 L_{\kappa\lambda}[g^1] = g_2 L_{\kappa\lambda}[g^2] = \dots = g_n L_{\kappa\lambda}[g^n], \quad (\kappa \neq \lambda).$$

C'est une condition nécessaire pour que les g^μ forment une métrique conforme-staeckelienne. Mais cette condition est aussi suffisante. Car, étant satisfaites les équations (12) par les g^μ , c'est-à-dire

$$(13) \quad g_{\mu} L_{\kappa\lambda}[g^{\mu}] = f_{\kappa\lambda}(x), \quad (\kappa \neq \lambda),$$

où les $f_{\kappa\lambda}$ sont des fonctions indépendantes de l'indice μ , les équations (11) s'écrivent

$$(14) \quad \frac{1}{\tau} L_{\kappa\lambda}[\tau] = f_{\kappa\lambda}(x), \quad (\kappa \neq \lambda).$$

Conformément à (13), $\tau = g^{\nu}$ ($\nu = 1, 2, \dots, n$) sont des solutions de (14). Ainsi nous avons le

Lemme 2. *Les n fonctions $g^{\mu}(x) \neq 0$ forment une métrique coforme-staeckelienne si, et seulement si, elles satisfont aux équations différentielles*

$$(12) \quad g_1 L_{\kappa\lambda}[g^1] = g_2 L_{\kappa\lambda}[g^2] = \dots = g_n L_{\kappa\lambda}[g^n], \quad (\kappa \neq \lambda).$$

Les équations (12) remplacent la condition (B1) du théorème 1.

De plus, supposant les g^{μ} sous l'hypothèse (12), on peut donner la solution générale $\tau(x)$ des équations (11), à savoir

$$\tau(x) = \sum_{\nu=1}^n \psi_{\nu}(x^{\nu}) g^{\nu}(x),$$

où chaque $\psi_{\nu}(x^{\nu})$ est une fonction arbitraire d'une seule variable. En effet, ayant choisi les ψ_{ν} arbitrairement, on a $L_{\kappa\lambda}[\psi_i g^i] = \psi_i L_{\kappa\lambda}[g^i]$ (comme on peut vérifier aisément), et de là $L_{\kappa\lambda}[\tau] = \psi_i g^i f_{\kappa\lambda} = \tau f_{\kappa\lambda}$. Ainsi, $\tau = \psi_i g^i$ est une solution de (14). D'autre part, τ étant solution de (11), g^{μ} a la forme $g^{\mu} = \tau G^{\mu}$, où G^{μ} est du type de Staeckel. Donc, il y a n fonctions $\psi_{\nu}(x^{\nu})$ tel qu'on a $1 = \psi_i G^i$ (première équation de (3)), et de là $\tau = \psi_i \tau G^i = \psi_i g^i$.

Lemme 3. *L'ensemble des métriques staeckeliennes adjointes à une métrique orthogonale g^{μ} du type conforme-staeckelien est donné par*

$$(15) \quad G^{\mu} = \tau g^{\mu}, \quad \tau(x) = \psi_i g^i, \quad \psi_{\nu} = \psi_{\nu}(x^{\nu}) \text{ arbitraire}$$

de sorte que $\tau \neq 0$.

Maintenant, nous nous occupons de la condition (8'). Là, nous éliminons les fonctions h_{μ} en prenant d'abord le logarithme et ensuite dérivant deux fois par rapport aux x^{κ} et x^{λ} ($\kappa \neq \lambda$). Puis, nous avons la condition équivalente

$$(16) \quad (n-2) \sigma_{\kappa\lambda} = [\log |\psi| G|^{-1/2}|]_{,\kappa\lambda} \quad (\kappa \neq \lambda),$$

ayant abrégé $\prod_{\nu=1}^n G_{\nu} = G$. Acceptons de [1] le lemme suivant :

Lemme 4. *Si la métrique G^{μ} est du type de Staeckel, on a pour le tenseur de Ricci de cette métrique*

$$(17) \quad \bar{R}_{\kappa\lambda} = \frac{3}{2} [\log |\psi^{-1}| G|^{1/2}|]_{,\kappa\lambda} \quad (\kappa \neq \lambda).$$

Avec (16) et ce lemme, nous arrivons à la forme suivante de la condition (C 1),

$$(18) \quad (n-2) \sigma_{\kappa\lambda} = -\frac{2}{3} \bar{R}_{\kappa\lambda}, \quad (\kappa \neq \lambda).$$

Cette condition figure déjà dans [1]. Mais, parce que la métrique G^μ est pratiquement inconnue, il faut transformer les composantes $\bar{R}_{\kappa\lambda}$ de son tenseur de Ricci et les exprimer par les composantes $R_{\kappa\lambda}$ du tenseur de Ricci qui est défini par la métrique g^μ donnée.

À cette fin, nous rappelons une formule de la géométrie riemannienne, [4]. Sous une transformation conforme de la métrique, $g_{ij} \rightarrow \bar{g}_{ij} = e^{2\rho} g_{ij}$, le tenseur de Riemman-Christoffel se transforme de manière que $\bar{R}_{lkij} = e^{2\rho} [R_{lkij} + g_{lj} S_{ki} + g_{ki} S_{lj} - g_{li} S_{kj} - g_{kj} S_{li}]$, où $S_{ij} \equiv \nabla_i \rho_j - \rho_i \rho_j + \frac{1}{2} g_{ij} \rho_r \rho^r$, $\rho_i \equiv \rho_{,i}$. D'ici, il s'en suit pour le tenseur de Ricci $\bar{R}_{ki} \equiv \bar{g}^{lj} \bar{R}_{lkij} = e^{-2\rho} g^{lj} \bar{R}_{lkij} = R_{ki} + (n-2) S_{ki} + g_{ki} S^l_l$.

Dans notre cas, $G^\mu = e^{2\tau} g^\mu$ suivant (7₁), il reste à identifier $\rho = -\sigma$ et $\bar{g}^{\kappa\lambda} = \delta^\kappa G^\lambda$, $g^{\kappa\lambda} = \delta^\kappa g^\lambda$, donnant

$$\bar{R}_{\kappa\lambda} = R_{\kappa\lambda} - (n-2) [\nabla_\kappa \sigma_{,\lambda} + \sigma_{,\kappa} \sigma_{,\lambda}], \quad (\kappa \neq \lambda).$$

Par le calcul de la dérivée covariante dans les crochets (utilisant les expressions explicites des symboles de Christoffel en coordonnées orthogonales), nous obtenons

$$\bar{R}_{\kappa\lambda} = R_{\kappa\lambda} - (n-2) [\sigma_{,\kappa\lambda} + \frac{1}{2} g_{\kappa\lambda} g_{,\lambda} \sigma_{,\kappa} + \frac{1}{2} g_{\lambda\kappa} g_{,\kappa} \sigma_{,\lambda} + \sigma_{,\kappa} \sigma_{,\lambda}], \quad (\kappa \neq \lambda)$$

et, introduisant l'opérateur $L_{\kappa\lambda}$ défini par (4),

$$(19) \quad \bar{R} = R_{\kappa\lambda} - \frac{n-2}{2} [L_{\kappa\lambda}[\sigma] - 3\sigma_{,\kappa\lambda} - 2\sigma_{,\kappa} \sigma_{,\lambda}], \quad (\kappa \neq \lambda).$$

À l'aide de cette formule, nous pouvons présenter la condition (18) sous la forme nouvelle

$$(20) \quad \frac{n-2}{4} \{L_{\kappa\lambda}[\sigma] - 2\sigma_{,\kappa} \sigma_{,\lambda}\} = -R_{\kappa\lambda}, \quad (\kappa \neq \lambda).$$

Enfin, faisons de nouveau la substitution $\tau = e^{-2\sigma}$ et, après des calculs simples, nous obtenons

$$(21) \quad \frac{n-2}{4} L_{\kappa\lambda}[\tau] = \tau R_{\kappa\lambda}, \quad (\kappa \neq \lambda),$$

comme forme transformée de la condition (C 1) du théorème 1.

Les relations (11) et (21) donnent lieu à la forme suivante du théorème de séparabilité équivalente au théorème 1 :

Théorème 2. *Pour que l'équation (1) soit séparable, il faut, et il suffit, que les conditions suivantes soient réalisées:*

(A 2) *Les coordonnées sont orthogonales :* $g^{\kappa\lambda} = \delta_{\lambda}^{\kappa} g^{\kappa}$.

(B 2) *Il existe une fonction $\tau(x) \neq 0$ satisfaisant aux équations différentielles (11). et*

$$(21) \quad \frac{n-2}{4} L_{\kappa\lambda}[\tau] = \tau R_{\kappa\lambda}, \quad (\kappa \neq \lambda).$$

La condition (B 2) montre que la métrique g^{μ} est du type conforme-staekelien avec un facteur de conformité τ qui satisfait à l'équation (21).

À cause du facteur $n-2$, il faut considérer séparément les cas $n > 2$ et $n = 2$.

5. Dimension $n > 2$

Par combinaison linéaire convenable, la condition (B 2) peut être remplacée par :

(B 2') *Les g^{μ} satisfont aux équations*

$$(22) \quad L_{\kappa\lambda}[g^{\mu}] = \frac{4}{n-2} g^{\mu} R_{\kappa\lambda}, \quad (\kappa \neq \lambda)$$

et il existe une telle fonction $\tau(x) \neq 0$ que

$$L_{\kappa\lambda}[\tau] = \frac{4}{n-2} \tau R_{\kappa\lambda}, \quad (\kappa \neq \lambda).$$

De (22), il s'en suit que les g^{μ} font partie d'une classe spéciale de métriques conforme-staekeliennes, car, au lieu de (13), nous avons ici

$$(24) \quad g_{\mu} L_{\kappa\lambda}[g^{\mu}] = f_{\kappa\lambda}(x) \quad \text{avec} \quad f_{\kappa\lambda}(x) = \frac{4}{n-2} R_{\kappa\lambda}, \quad (\kappa \neq \lambda).$$

Or, si les g^{μ} satisfont aux équations (24), les équations (23) ont la solution générale $\tau = \psi_i g^i$, $\psi_v = \psi_v(x^v)$ arbitrairement choisi, comme nous savons de la démonstration du lemme 3. Ainsi, le remplissement de (22) suffit pour la résolubilité de (23) et, par conséquent, nous arrivons finalement au

Théorème 3. *Pour que l'équation (1) à $n \geq 3$ variables soit séparable, il faut, et il suffit, que :*

(A 3) *La métrique soit orthogonale :* $g^{\kappa\lambda} = \delta_{\lambda}^{\kappa} g^{\kappa}$.

(B 3) *Les fonctions g^{μ} satisfassent aux équations différentielles (22), où l'opérateur $L_{\kappa\lambda}$ du 2^{ème} ordre est donné par (4) et où $R_{\kappa\lambda}$ désigne le tenseur de Ricci adjoint à la métrique g^{μ} .*

Nous donnons encore une autre forme de ce théorème, qui est plus commode pour l'application. Mais d'abord, nous obtenons immédiatement deux conclusions remarquables.

Théorème 4. *Si la métrique g^{ij} dans l'équation (1) à $n \geq 3$ variables est orthogonale et du type de Staeckel¹, il faut, et il suffit, pour la séparabilité de cette équation, que le tenseur de Ricci ait la forme diagonale,*

$$(22') \quad R_{\kappa\lambda} = 0, \quad (\kappa \neq \lambda).$$

Car, g^μ étant du type staeckelien, nous avons $L_{\kappa\lambda}[g^\mu] = 0$ ($\kappa \neq \lambda$) et par là, la condition (22) prend la forme (22'), qui est connue comme condition de Robertson dans la théorie de séparation de l'équation de Schroedinger [5].

Théorème 5. *Dans les espaces d'Einstein (renfermant les espaces euclidiens) il faut, et il suffit, pour la séparabilité de l'équation (1) à $n \geq 3$ variables, que la métrique g^{ij} soit orthogonale et du type de Staackel.*

Les espaces d'Einstein sont caractérisés par la relation invariante $R^{ij} = \frac{1}{n} R g^{ij}$ ($R \equiv R_i^i$: courbure scalaire). Or, en coordonnées orthogonales, il s'en suit de cette relation que $R_{\kappa\lambda} = 0$ ($\kappa \neq \lambda$) et donc, par (22), nous avons $L_{\kappa\lambda}[g^\mu] = 0$ ($\kappa \neq \lambda$) comme condition restante.

En coordonnées orthogonales, on a pour le tenseur de Ricci

$$(25) \quad R_{\kappa\lambda} \equiv \frac{1}{4} \sum_{i \neq \kappa, \lambda} \{g_i L_{\kappa\lambda}[g^i] - 3(\log |g^i|)_{,\kappa\lambda}\}, \quad (\kappa \neq \lambda),$$

comme on peut le calculer en partant de la définition de ce tenseur. Soit satisfaite l'équation (22). Puis, il s'en suit de (12) : $g_1 L_{\kappa\lambda}[g^1] = \dots = g_n L_{\kappa\lambda}[g^n]$ et de plus, par sommation, $\sum_{\mu \neq \kappa, \lambda} g_\mu L_{\kappa\lambda}[g^\mu] = 4R_{\kappa\lambda}$.

Ainsi, de (25)

$$(26) \quad \sum_{i \neq \kappa, \lambda} (\log |g^i|)_{,\kappa\lambda} = 0, \quad (\kappa \neq \lambda).$$

D'autre part, avec (25) et (12), nous avons par (25)

$$R_{\kappa\lambda} = \frac{1}{4} \sum_{i \neq \kappa, \lambda} g_i L_{\kappa\lambda}[g^i] = \frac{n-2}{4} g_\mu L_{\kappa\lambda}[g^\mu], \quad (\kappa \neq \alpha),$$

pour chaque indice μ c'est-à-dire la condition (22) est satisfaite.

Observons encore, que $\sum_{i \neq \kappa, \lambda} \log |g^i| \equiv \log |g_\kappa g_\lambda \prod_{i=1}^n g^i| \equiv -\log |g^\kappa g^\lambda g|$. Donc

(26) montre que les variables x^κ, x^λ sont séparées additivement dans le dernier logarithme, c'est-à-dire $g^\kappa g^\lambda g$ a la structure $g^\kappa g^\lambda g = F^{\kappa\lambda}$ (non x^κ) $\times$

$\times F^{\lambda\kappa}$ (non x^λ). D'ici, nous pouvons donner le théorème 3 sous la nouvelle forme :

Théorème 6. *Pour que l'équation (1) à $n \geq 3$ variables soit séparable, il faut, et il suffit, que :*

(A 6) *La métrique soit orthogonale : $g^{\kappa\lambda} = \delta_\lambda^\kappa g^\kappa$.*

(B 6) *Les fonctions g^μ satisfassent aux équations différentielles (12).*

(C 6) *Les produits $g^\kappa g^\lambda g \equiv \prod_{i \neq \kappa, \lambda} g_i$ aient la structure*

$$(27) \quad g^\kappa g^\lambda g = F^{\kappa\lambda} (\text{non } x^\kappa) F^{\lambda\kappa} (\text{non } x^\lambda), \quad (\kappa \neq \lambda),$$

où les fonctions F ne dépendent pas de la variable indiquée.

À l'aide de ce théorème, on peut décider si une équation (1) avec des fonctions g^{ij} données est séparable ou non, en effectuant seulement quelques calculs simples. L'ensemble des théorèmes 1 et 6 répond au problème de séparabilité de l'équation de Laplace d'une manière complète. Les équations séparées associées se trouvent dans [1].

6. Dimension $n = 2$

Partons du théorème 2. Là, la condition (21) est satisfaite identiquement à cause du facteur $n - 2$ et en vue de (25). Il reste la condition (11), qui est (voir lemme 2) équivalente à $g_1 L_{12}[g^1] = g_2 L_{12}[g^2]$. Après quelques calculs simples, cette condition prend la forme $(\log |g^1 g^2|)_{,12} = 0$ et donc, g^1 et g^2 satisfont la relation

$$\frac{g^1}{g^2} = \frac{h_1(x^1)}{h_2(x^2)},$$

avec des fonctions h arbitraires. Nous avons le

Théorème 7. *L'équation (1) à deux variables si, et seulement si :*

(A 7) *La métrique est orthogonale.*

(B 7) *Les fonctions g^μ sont de la forme*

$$(28) \quad g^1 = h(x^1, x^2) h_1(x^1), \quad g^2 = h(x^1, x^2) h_2(x^2),$$

où les fonctions h, h_1, h_2 sont arbitraires et non-nulles.

En vue de (28), la métrique est donc du type général conforme-euclidien.

7. Exemples

Soit toujours $n > 2$ en ce qui suit.

7.1. Considérons le cas des coordonnées isothermes en espace V_n (qui doit être du type conforme-euclidien),

$$(29) \quad g^\mu = \varepsilon_\mu h(x), \quad h > 0, \quad \varepsilon_\mu = \pm 1.$$

Naturellement, la condition (12) du théorème 6 est satisfaite, et la condition (27) donne

$$(30) \quad h(x) = \prod_{\nu=1}^n h_{\nu}(x^{\nu}), \quad h_{\nu}(x^{\nu}) \text{ arbitraire}$$

comme caractérisation du cas séparable.

Si la métrique (29) est de plus du type de Liouville: $h(x) = \left[\sum_{\lambda=1}^n f_{\lambda}(x^{\lambda}) \right]^{-1}$, il s'en suit de (30) comme condition de séparabilité: il n'y a tout au plus qu'une des fonctions f_{λ} qui est non-constante.

7.2. Soit une des fonctions g^{μ} constante, par exemple $g^n = \text{const.}$ Il résulte du théorème 3 (en mettant $\mu = n$ dans (22)) que les g^{μ} doivent être du type de Staeckel et doivent satisfaire à la condition de Robertson $R_{\kappa\lambda} = 0$ ($\kappa \neq \lambda$).

D'ici, nous obtenons immédiatement les conditions de séparabilité pour l'équation ondulatoire

$$(31) \quad 0 = \Delta u \equiv \left[\Delta' - \left(\frac{\partial}{\partial x^n} \right)^2 \right] u,$$

où Δ' désigne l'opérateur de Laplace dans l'hypersurface $x^n = 0$, ayant une métrique $g_{ij}(x^1, \dots, x^{n-1})$, ($i, j = 1, \dots, n-1$), positive définie:

$$(32) \quad \left. \begin{aligned} g^{\kappa\lambda} &= \delta^{\kappa\lambda} g^{\kappa} \\ L_{\kappa\lambda}[g^{\mu}] &= 0 \quad (\kappa \neq \lambda) \\ R_{\kappa\lambda} &= 0 \quad (\kappa \neq \lambda) \end{aligned} \right\} (\mu, \kappa, \lambda = 1, \dots, n-1).$$

Ce sont exactement les conditions de séparabilité pour l'équation de Schroedinger à $n-1$ dimensions et avec un potentiel constant. Ce résultat est évident, car, dans l'équation (31), nous pouvons séparer la variable x^n sous la forme $u(x) = v(x^1, \dots, x^{n-1})w(x^n)$, arrivant à une équation de Helmholtz $\Delta'v + \alpha v = 0$ (équation Schroedinger particulière).

7.3. Considérons la métrique orthogonale tridimensionnelle

$$(33) \quad g^{\kappa} = \left[\prod_{\substack{k=1 \\ k \neq \kappa}}^3 (\varphi_k - \varphi) \right]^{-1}, \quad (\kappa = 1, 2, 3),$$

avec des fonctions $\varphi_{\nu} = \varphi_{\nu}(x^{\nu})$ données (soit $\varphi_{\nu} \neq \varphi_{\mu}$ ($\nu \neq \mu$) partout dans le domaine considéré). Cette métrique (à n arbitraire) a été donnée par Levi-Civita dans la théorie de la correspondance des systèmes mécaniques. Elle donne lieu à l'équation de Laplace

$$(34) \quad (\varphi_2 - \varphi_3)u_{,11} + (\varphi_3 - \varphi_1)u_{,22} + (\varphi_1 - \varphi_2)u_{,33} = 0.$$

La métrique (33) est du type de Staeckel (à vérifier par lemme 1) et, de plus, elle satisfait aux équations (27). Ainsi, l'équation (34) est séparable; les équations séparées sont

$$(34') \quad u_{,\nu}^2 + (\alpha_1 \varphi_{\nu} + \alpha_2)u_{,\nu} = 0.$$

7.4. Finissons nos considérations par l'exemple de la métrique orthogonale

$$(35) \quad g^1 = \varphi_2 - \varphi_3, \quad g^2 = \varphi_3 - \varphi_1, \quad g^3 = \varphi_1 - \varphi_2; \quad \varphi_{\nu} = \varphi_{\nu}(x^{\nu}).$$

Cette métrique est du type conforme-staeckelien, comme on vérifie par le lemme 2 ou en observant qu'elle est conforme à la métrique staeckelienne (33) avec le facteur de conformité $\tau = -(\varphi_1 - \varphi_2)(\varphi_2 - \varphi_3)(\varphi_3 - \varphi_1)$. Elle est du type de Staeckel si, au moins deux des fonctions φ_i sont constantes. Abandonnant ce cas dégénéré, l'équation de Laplace adjointe sera inséparable, car on trouve $g^1 g^2 g = (\varphi_1 - \varphi_2)^{-1}$, etc., et donc, la condition (27) du théorème 6 n'est pas satisfaite.

Reçue le 1 juin 1968

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Thermodynamique,
République Démocrate Allemande

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ASUPRA CONDIȚIILOR DE SEPARABILITATE A VARIABILELOR PENTRU ECUAȚIA LAPLACE ÎN SPAȚIUL RIEMANN

(Rezumat)

Plecînd de la condițiile de separabilitate date de Ober-Kassenbaum pentru ecuații de tip Laplace, se dau condiții echivalente care permit, grație unor calcule simple, să se decidă dacă o ecuație dată de acest tip este sau nu separabilă.

SERIES REPRESENTATIONS FOR DUAL LAGUERRE TEMPERATURES¹⁾

BY

DEBORAH TEPPER HAIMO and FRANK M. CHOLEWINSKI

1. It is our goal in this Note to derive criteria for the representation of solutions of the Laguerre differential heat equation

$$(1) \quad xu_{xx} + (\alpha + 1 - x)u_x = u_t$$

in series of Laguerre heat polynomials $p_{n,\alpha}(x,t)$, where

$$(2) \quad p_{n,\alpha}(x,t) = \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} (xe^{-t})^k (1-e^{-t})^{n-k},$$

and in series of dual Laguerre temperature transforms $w_{n,\alpha}(x,t)$ of $p_{n,\alpha}(x,t)$, where

$$(3) \quad w_{n,\alpha}(x,t) = g(x;t) p_{n,\alpha} \left[\frac{x}{e^t - 1}, (1 - e^{-t}) \right], \quad t \geq 0,$$

$g(x;t) = g(x,0;t)$ being the fundamental solution of (1) with

$$(4) \quad g(x,y;t) = \left(\frac{e^t}{e^t - 1} \right)^{\alpha+1} e^{-\frac{x+y}{e^t-1}} \vartheta \left(\frac{2\sqrt{xye^t}}{e^t - 1} \right), \quad t > 0.$$

Here $\vartheta(z) = 2^\alpha \Gamma(\alpha+1) z^{-\alpha} I_\alpha(z)$, where $I_\alpha(z)$ is the Bessel function of imaginary argument of order α . The results will be stated without proofs, full details to appear in another publication [8]. The theory is analogous to that developed by D. T. Haimo for the generalized heat equation

$$u_{xx} + \frac{2v}{x} u_x = u_t$$

¹⁾ The research of the first author was supported in part by the National Science Foundation under grant $\neq$ GP 12092 and by the University of Missouri—St. Louis and that of the second author by the National Science Foundation under grant $\neq$ GP 13260.

in [2], the latter extending the work of P. C. Rosenbloom and D. V. Widder in [9] for the ordinary heat equation $u_{xx} = u_t$.

2. A dual Laguerre temperature is defined as a C^2 solution of (1). The class of all dual Laguerre temperatures will be denoted by H .

A subclass H^* of H for $a < t < b$ is the class of all dual Laguerre temperatures $u(x, t)$ such that for all $t, t', a < t' < t < b$,

$$(5) \quad u(x, t) = \int_0^\infty g(x, y; t - t') u(y, t') d\wedge(y),$$

where

$$d\wedge(y) = \frac{1}{\Gamma(\alpha + 1)} y^\alpha e^{-y} dy,$$

the integral converging for $a < t < b$. We designate by H_A^* the subclass of H^* for which the integral of (5) converges absolutely.

The dual Laguerre temperature transform $u^T(x, t)$ of $u(x, t) \in H$ is given by

$$(6) \quad u^T(x, t) = g(x; t) u\left(\frac{x}{e^t - 1}, \ln(1 - e^{-t})\right), \quad t > 0.$$

It follows that if $u(x, t) \in H$ for $a < t < b \leq 0$ then $u^T(x, t) \in H$ for $\ln[1/(1 - e^b)] > t > \ln[1/(1 - e^a)]$.

The Laguerre heat polynomials $p_{n,\alpha}(x, t) \in H_A^*$ for all t , whereas $w_{n,\alpha}(x, t) = p_{n,\alpha}^T(x, t)$, $t > 0$, $\in H_A^*$ for $t > 0$.

An important property of the $p_{n,\alpha}(x, t)$ and their transforms $w_{n,\alpha}(x, t)$ is their biorthogonality in the sense that

$$(7) \quad \int_0^\infty p_{n,\alpha}(x, -t) x_{n,\alpha}(x, t) d\wedge(x) = n! \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} \delta_{nn}.$$

In addition, these functions satisfy the following fundamental relation for $|t| < s$, $s > 0$,

$$(8) \quad \sum_{n=0}^\infty \frac{\Gamma(\alpha + 1)}{n! \Gamma(n + \alpha + 1)} p_{n,\alpha}(x, t) w_{n,\alpha}(y, s) = g(x, y; s + t).$$

By an appeal to asymptotic estimates of $p_{n,\alpha}(x, t)$, we may establish that the region of convergence of the series of Laguerre heat polynomials is generally a non-symmetric strip about the axis $t = 0$ or else is an upper half plane with axis below $t = 0$ as states in the following result.

Theorem 1. *If, for $\gamma \geq 0$, $\lim_{n \rightarrow \infty} \frac{n}{e} |a_n|^{1/n} = \gamma$, then the series*

$$\sum_{n=0}^{\infty} a_n p_{n,\alpha}(x, t)$$

converges absolutely for $\ln[\gamma/(1+\gamma)] < t < \infty$ if $0 \leq \gamma \leq 1$, and $\ln[\gamma/(1+\gamma)] < t < \ln[\gamma/(\gamma-1)]$ if $\gamma > 1$.

In a similar way, it can be shown that the corresponding series of dual Laguerre temperature transforms converges in an upper half plane with axis above $t = 0$ as given in the following

Theorem 2. *If, for $\gamma \geq 0$, $\lim_{n \rightarrow \infty} \frac{n}{e} |b_n|^{1/n} = \gamma$, then the series*

$$\sum_{n=0}^{\infty} b_n w_{n,\alpha}(x, t)$$

converges absolutely for $0 \leq \ln(1+\gamma) < t$.

Within their regions of convergence, both the series $\sum_{n=0}^{\infty} a_n p_{n,\alpha}(x, t)$

and $\sum_{n=0}^{\infty} b_n w_{n,\alpha}(x, t)$ represent dual Laguerre temperatures.

Membership in H_A^* of the $p_{n,\alpha}(x, t)$, the generating function (8), and asymptotic estimates of $p_{n,\alpha}(x, t)$ and $w_{n,\alpha}(x, t)$ are factors essential in the establishment of the following principal expansion theorem in series of Laguerre heat polynomials.

Theorem 3. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} a_n p_{n,\alpha}(x, t),$$

the series converging for $\ln[\gamma/(\gamma+1)] < t < \ln[\gamma/(\gamma-1)]$ if $\gamma > 1$, $\ln[\gamma/(\gamma+1)] < t < \infty$ if $0 \leq \gamma \leq 1$, is that $u(x, t) \in H_A^$ there. The coefficients a_n have either of the determinations $a_n = u^{(n)}(0, 0)/n!$, or*

$$a_n = \frac{\Gamma(\alpha+1)}{n! \Gamma(n+\alpha+1)} \int_0^{\infty} u(y, -t) w_{n,\alpha}(y, t) d\wedge(y), \quad 0 < t < \ln \frac{\gamma+1}{\gamma}.$$

It can be shown that the integral defining the coefficient a_n is a constant independent of t .

The theorem may be illustrated by the function

$$u(x, t) = e^{a(1-e^{-t})} \wp(2\sqrt{axe^{-t}}),$$

which is a member of H_A^* for all t and has the expansion

$$u(x, t) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 1)}{n! \Gamma(n + \alpha + 1)} a^n p_{n, \alpha}(x, t).$$

The coefficients here have the desired integral representation since

$$a^n = \int_0^{\infty} e^{a(1-e^t)} (2\sqrt{axe^t}) w_{n, \alpha}(y, t) d\wedge(y).$$

As an immediate consequence of the definitions, the following expansion theorem for series of the dual Laguerre temperature transforms may be proved:

Theorem 4. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} b_n w_{n, \alpha}(x, t),$$

the series converging for $t > \ln(\gamma + 1)$, $\gamma \geq 0$, is that there exist a function $v(x, t)$ in H_A^ for $\ln[\gamma/(\gamma + 1)] < t < \ln[\gamma/(\gamma - 1)]$ if $\gamma \geq 1$, $\ln[\gamma/(\gamma + 1)] < t < \infty$ if $0 \leq \gamma \leq 1$, and such that $u(x, t) = v^T(x, t)$.*

The theorem may be illustrated by $u(x, t) = g(a, x; t)$.

Here $u(x, t) = v^T(x, t)$, where $v(x, t) = e^{a(1-e^{-t})} (2\sqrt{axe^{-t}})$ is in H_A^* for all t , and we have

$$u(x, t) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 1)}{n! \Gamma(n + \alpha + 1)} a^n w_{n, \alpha}(x, t), \quad t > 0.$$

Criteria of a different nature yield additional series representation theorems. To state these, we note that an analytic function $\varphi(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ is said to be of growth (ρ, τ) if and only if $\lim_{n \rightarrow \infty} \frac{n}{\rho} |a_n|^{1/n} \leq e \tau$.

We then have the following result for the Laguerre heat polynomials:

Theorem 5. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} a_n p_{n, \alpha}(x, t),$$

the series converging for $\ln[\gamma/(\gamma + 1)] < t < \ln[(\gamma/\gamma - 1)]$ if $\gamma \geq 1$, $\ln[\gamma/(\gamma + 1)] < t < \infty$ if $0 \leq \gamma \leq 1$, is that

$$u(x, t) = \int_0^{\infty} g(x, y; t) \varphi(y) d\wedge(y),$$

$0 < t < \ln[\gamma/(\gamma-1)]$ if $\alpha \geq 1$, $0 < t < \infty$ if $0 \leq \gamma \leq 1$, where φ is an entire function of growth $(1, \gamma)$ and $a_n = \varphi^{(n)}(0)/n!$.

Correspondingly, for the series of $w_{n,\alpha}(x, t)$ we have the following

Theorem 6. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} b_n w_{n,\alpha}(x, t),$$

the series converging for $t > \ln(\gamma + 1)$, $\gamma \geq 0$, is that

$$u(x, t) = (e^t)^{\alpha+1} e^x \int_0^{\infty} e^{-y(e^t-2)} J(2\sqrt{xye^t}) \varphi(y) d\wedge(y),$$

where φ is an entire function of growth $(1, \gamma)$ and $b_n = (-1)^n \varphi^{(n)}(0)/n!$.

Here $J(z) = 2^\alpha \Gamma(\alpha + 1) z^{-\alpha} J_\alpha(z)$ where $J_\alpha(z)$ is the ordinary Bessel function of order α .

Membership in H_A^* is not sufficient for the representation of the function $u(x, t)$ by the series $\sum_{n=0}^{\infty} b_n w_{n,\alpha}(x, t)$ as is indicated by the function $u(x, t) = (e^t)^{\alpha+1} e^x$. Needed is an additional integrability condition. The resulting theorem is established by an appeal to theorem 6, to asymptotic estimates of $p_{n,\alpha}(x, t)$ and $w_{n,\alpha}(x, t)$, and to the fact that a function $u(x, t)$ given by the series $\sum_{n=0}^{\infty} b_n w_{n,\alpha}(x, t)$ for $t > \ln(\gamma + 1) \geq 0$ has an upper bound

$$|u(x, t)| \leq M (e^t)^{\frac{\alpha}{2} + \frac{3}{4}} x^{-\frac{\alpha}{2} + \frac{1}{4}} \frac{e^{c-2}}{\sqrt{e^t - e^c}} e^{\frac{e^c-2}{e^t+e^c-2}x}, \quad t > c,$$

for every $c > \ln(\gamma + 1)$ where $M = M(c)$ is a positive number. Following is the principal result corresponding to Theorem 3.

Theorem 7. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} b_n w_{n,\alpha}(x, t),$$

the series converging for $t > \ln(\gamma + 1)$, $\gamma \geq 0$, is that $u(x, t) \in H_A^$ there and that*

$$\int_0^{\infty} e^{\frac{ye^t}{2(e^t-1)}} |u(y, t)| d\wedge(y) < \infty, \quad t > \ln(\gamma + 1).$$

The coefficients b_n have the representation

$$b_n = \frac{\Gamma(\alpha + 1)}{n! \Gamma(n + \alpha + 1)} \int_0^\infty u(y, t) p_{n, \alpha}(y, -t) d\Lambda(y), \quad t > \ln(\gamma + 1),$$

where the integral is a constant independent of t .

Received December 1, 1969

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REPREZENTĂRI PRIN SERII PENTRU POLINOAME LAGUERRE TERMICE DUALE

(Rezumat)

Se formulează câteva teoreme privind reprezentarea soluțiilor ecuației diferențiale (1) prin serii de polinoame Laguerre termice duale.

UNE NOUVELLE PRÉSENTATION GRAPHIQUE DE LA THÉORIE DES VECTEURS GLISSANTS

I. CARACTÉRISTIQUES VECTORIELLES PLANES-AXIELLES D'UN SYSTÈME DE VECTEURS GLISSANTS ¹⁾

PAR

C. HUIU, N. IRIMICIUC et GABRIELA NEAGU

Introduction

Le procédé graphique connu, tout simple, rapide et assez précis, de réduction des systèmes de vecteurs glissants coplanaires, basé sur la construction d'un polygone vectoriel et d'un polygone funiculaire, sera longtemps utilisé dans la pratique de l'ingénieur.

L'extension de tels procédés à la résolution des problèmes spatiaux de la théorie des vecteurs glissants a échoué, parce que :

a) toutes les méthodes proposées par les divers auteurs sont très compliquées et donc inopérantes ;

b) toutes ces méthodes diffèrent de celles utilisées pour le cas plan, ce qui ne permet pas une présentation graphique unitaire de la théorie des vecteurs glissants.

Un procédé graphique qui a éliminé substantiellement les inconvénients signalés ci-dessus a été élaboré par l'un des auteurs, [1], mais la méthode proposée, qui contient comme un cas particulier la présentation habituelle, plane, du problème, comporte encore un grand nombre de constructions graphiques auxiliaires.

Dans cette Note, tout en utilisant quelques nouvelles idées dans la représentation des objets spatiaux, incluses dans la *Géométrie représentative plane-axielle* élaborée par les deux premiers auteurs, [2] ... [4], on expose une nouvelle méthode graphique de réduction des systèmes de vecteurs glissants spatiaux.

¹⁾ Présentée lors de la session scientifique de l'Institut Polytechnique de Jassy, le 2 — 4 juillet 1970.

1. Détermination des vecteurs glissants

Soit un système de grandeurs vectorielles, $\mathbf{V}_\mu$ ($\mu = 1, 2, \dots, m$) représentées, toutes, à la même échelle:

$$(I.1) \quad v = \frac{v_V}{q_V} \left(\frac{u_V}{\text{cm}} \right),$$

v_V étant le nombre des unités de mesure, u_V , des grandeurs représentées par un segment de q_V cm longueur — par les vecteurs glissants $\mathbf{v}_\mu$ c'est-à-dire:

$$(I.2) \quad \mathbf{V}_\mu = \sigma_V \mathbf{v}_\mu, \quad (\mu = 1, 2, \dots, m).$$

Si:

$$(I.3) \quad \mathbf{R}_\mu = \sigma_L \mathbf{r}_\mu, \quad (\mu = 1, 2, \dots, m),$$

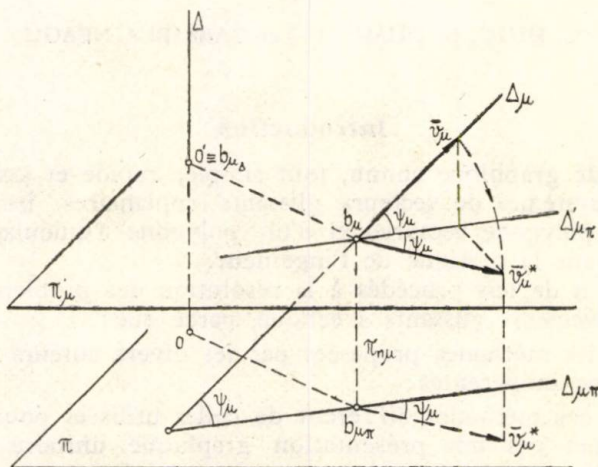


Fig. 1

sont les vecteurs de position par rapport à une origine commune, O , des points d'application des grandeurs vectorielles, représentés à leur tour à l'échelle

$$(I.4) \quad \sigma_L = \frac{u_L}{q_L} \left(\frac{\text{m}}{\text{cm}} \right)$$

par les vecteurs $\mathbf{r}_\mu$, les moments par rapport à l'origine choisie auront les expressions:

$$(I.5) \quad \mathbf{M}_\mu = \mathbf{R}_\mu \times \mathbf{V}_\mu = \sigma_L \sigma_V \mathbf{r}_\mu \times \mathbf{v}_\mu = \sigma_M \mathbf{m}_\mu,$$

où

$$(I.6) \quad \sigma_M = \sigma_L \sigma_V 1 \text{ cm} = \frac{v_L}{q_L} \frac{v_V}{q_V} \left(\frac{m_{\mu V}}{m} \right)$$

est l'échelle de représentation du moment M_μ par le vecteur m_μ .

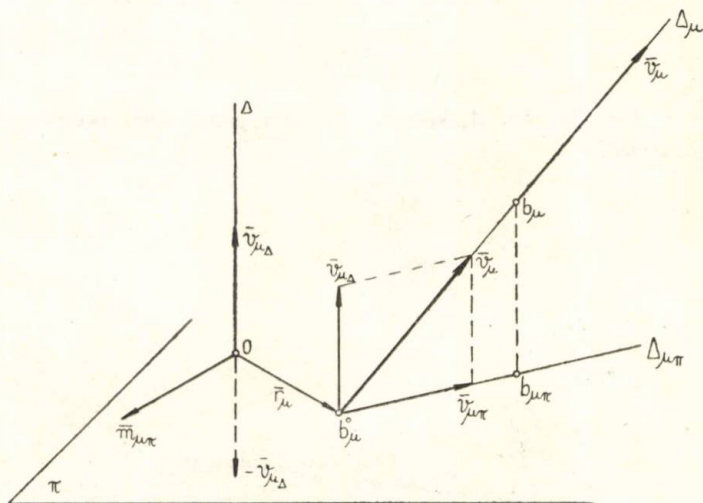


Fig. 2

Dans ce qui suit sont considérés connus, pour chaque vecteur du système, le support Δ_μ et le module et l'orientation du vecteur v_μ (fig. 1). Utilisant, comme dans [5], le plan de la représentation Π et l'axe de la représentation Δ , normale au plan, on voit aisément que le vecteur glissant v_μ , décomposé dans le point b_μ^0 d'intersection du support Δ_μ avec le plan Π (fig. 2), dans les composantes orthogonales $v_\mu = v_{\mu\Pi} + v_{\mu\Delta}$, sera complètement déterminé par les éléments suivants:

1° le vecteur glissant $v_{\mu\Pi}$ situé sur le support $\Delta_{\mu\Pi}$ — la projection sur le plan Π du support Δ_μ ;

2° le vecteur axial $v_{\mu\Delta}$, appliqué dans l'origine O et rabattu, de même que l'axe Δ , sur le plan Π (fig. 3);

3° le moment plan $m_{\mu\Pi}$ de la composante $v_{\mu\Delta}$, considéré par rapport à l'origine O , normale au vecteur $r_{\mu}^{b0} = r_\mu$ et satisfaisant la relation:

$$(I.7) \quad m_{\mu\Pi} 1 \text{ cm} = r_\mu \times v_{\mu\Delta}.$$

La détermination de tous ces éléments est indiquée dans la fig. 3 ($\Delta_{\mu\Pi}$ est la projection sur le plan Π du support Δ_μ , $\bar{v}_\mu^*$ est le vecteur $\bar{v}_\mu$ rabattu dans ce plan; $\bar{b}_{\mu\Pi}^* b_\mu^* = \bar{O} b_{\mu\Delta}$, $\bar{b}_\mu^* b_\mu^0 \parallel v_\mu^*$, $\bar{O} a_\mu = 1 \text{ cm}$, $\bar{b}_\mu^0 \gamma_\mu \parallel \alpha_\mu \lambda_\mu$, $\bar{O} \gamma_\mu = m_{\mu\Pi}$).

Les vecteurs $\mathbf{v}_{\mu\Delta}$, $\mathbf{v}_{\mu\Delta}$ et $\mathbf{m}_{\mu\Delta}$, qui déterminent complètement le vecteur glissant $\mathbf{v}_{\mu}$:

$$(I.8) \quad \mathbf{v}_{\mu} \left\{ \begin{array}{l} \mathbf{v}_{\mu\Delta} \\ \mathbf{v}_{\mu\Delta} \\ \mathbf{m}_{\mu\Delta} \end{array} \right\},$$

comme on voit dans la fig. 4, seront nommés *coordonnées vectorielles plane-axielles* du vecteur $\mathbf{v}_{\mu}$.

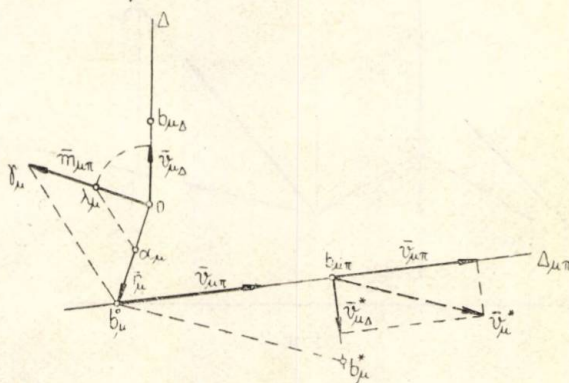


Fig. 3

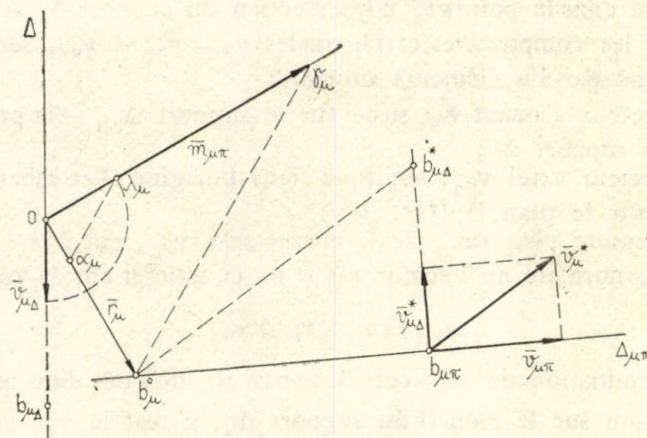


Fig. 4

II. Caractéristiques planes-axielles d'un système de vecteurs glissants

On introduit les notions suivantes concernant un système de vecteurs glissants $\mathbf{v}_\mu$ ($\mu = 1, 2, \dots, m$), chaque vecteur étant déterminé par les coordonnées vectorielles plane-axielles (I.8) :

a) la notion de *vecteur résultant axiel* définie par la relation

$$(II.1) \quad \mathbf{v}_\Delta = \sum_{\mu} \mathbf{v}_{\mu\Delta},$$

vecteur déterminé par la construction d'une ligne vectorielle;

b) la notion de *vecteur résultant plan*, définie par les relations

$$(II.2) \quad \begin{cases} \mathbf{v}_{\Pi} = \sum_{\mu} \mathbf{v}_{\mu\Pi}, \\ \mathbf{M}_0(\mathbf{v}_{\Pi}) = \sum_{\mu} \mathbf{M}_0(\mathbf{v}_{\mu\Pi}), \end{cases}$$

et déterminé par la construction d'un polygone vecteuriel et d'un polygone funiculaire dans le plan Π ;

c) la notion de *moment résultant plan*, définie par la relation

$$(II.3) \quad \mathbf{m}_{\Pi} = \sum_{\mu} \mathbf{m}_{\mu\Pi};$$

ce vecteur, appliqué aussi dans l'origine O , est déterminé par la construction d'un polygone vecteuriel.

Il est évident que le vecteur $\mathbf{m}_{\Pi}$ sera, d'après le théorème de Varignon, le moment de la résultante unique des vecteurs axiels $\mathbf{v}_{\mu\Delta}$ appliqués dans les points b_{μ}^0 , c'est-à-dire qu'on pourra écrire

$$(II.4) \quad \mathbf{m}_{\Pi} \text{ 1 cm} = \mathbf{r}_{b_{\Delta}^0} \times \mathbf{v}_{\Delta},$$

b_{Δ}^0 étant le point d'application du vecteur résultant $\mathbf{v}_{\Delta}$ situé dans le plan Π , sans être situé, en général, sur le support du vecteur résultant plan $\mathbf{v}_{\Pi}$.

Les vecteurs

$$(II.5) \quad \begin{Bmatrix} \mathbf{v}_{\Delta} \\ \mathbf{v}_{\Pi} \\ \mathbf{m}_{\Pi} \end{Bmatrix}$$

seront nommés *composantes planes-axielles* du torseur dans le pôle O du système des vecteurs $\mathbf{v}_{\mu}$.

Si l'on passe du pôle O à un autre pôle Q , situé aussi dans le plan Π , seulement le moment résultant plan supportera des modifications, conformément à la relation :

b) le vecteur de position du point O par rapport à l'origine O doit satisfaire la relation :

$$(II.10) \quad (\mathbf{m}_{II} - \mathbf{m}_{C_{II}}) \cdot 1 \text{ cm} = \mathbf{r}_C \times \mathbf{v}_\Delta.$$

Dans la fig. 7 est présentée la construction de l'axe centrale d'un système de vecteurs $\mathbf{v}_\mu$, lorsqu'on connaît les caractéristiques planes-axiales du système $\mathbf{v}_{II}$, $\mathbf{v}_\Delta$ et $\mathbf{m}_{II}$ ($\gamma\varepsilon = \mathbf{m}_{II} - \mathbf{m}_{C_{II}}$, $OC_0 \perp \gamma\varepsilon$, $d^0C_0 \perp \Delta_{II}$; $\Delta_{C_{II}} \parallel \Delta_{II}$ est la projection sur le plan Π de l'axe centrale et Δ_C^* — l'axe centrale rabattue dans le plan Π).

Reçu le 30 novembre 1970

Institut Politechnique, Jassy,
Chaire de Géométrie descriptive
et
Chaire de Mécanique théorique

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O NOUĂ PREZENTARE GRAFICĂ A TEORIEI VECTORILOR ALUNECĂTORI

(Rezumat)

Definindu-se, pentru determinarea unui vector alunecător spațial, noțiunile de *coordonate vectoriale plan-axiale*, reprezentate prin două componente ortogonale ale vectorului după plan și o axă rabătuț în plan și printr-un moment plan cu o orientare și mărime determinată, în lucrare se prezintă o nouă metodă grafică — denumită de autori *metoda coordonatelor vectoriale plan-axiale* — de prezentare a teoriei vectorilor alunecători.

În această primă notă se introduce noțiunile de *componente plan-axiale* ale torsiului într-un pol O al unui sistem de vectori alunecători și se arată posibilitatea de determinare pe cale grafică a axei centrale a sistemului.

ASUPRA ADAPTĂRII PROIECȚIEI CENTRALE ÎN SISTEMUL
DE REFERINȚĂ PLAN-AXIAL¹⁾

DE

I. RUSU

1. Introducere

În lucrări precedente, utilizînd sistemul de referință plan axial, s-a construit proiecția paralelă a unui obiect dat pe planul $P(P_0, P_z)$. În acest sens s-a considerat direcția de proiecție ortogonală, iar proiecția obiectului pe planul P s-a construit pe baza corespondenței afine dintre aceasta și proiecția aceluiași obiect pe planul Π .

În lucrarea de față, centrul de proiecție este punct propriu, deci proiecția obiectului pe planul P este conică, numită și perspectiva figurii sau corpului geometric ce au fost considerate.

Lucrarea prezintă rezultatele aplicării corespondenței perspective a figurilor plane, adaptată la sistemul de referință plan-axial.

2. Perspectiva punctului

Se consideră conul vizual proiectat pe planul Π și pe axa Oz .

Secționarea acestui con cu un plan $P(P_0, P_z)$ conduce la obținerea proiecției centrale (conice) a obiectului dat, sau perspectiva acestuia.

Astfel, raza vizuală $SA(sa, \bar{sa})$, fig. 1, este intersectată cu planul considerat $P(P_0, P_z)$ în punctul $A_1(a_1, \bar{a}_1)$, care este perspectiva punctului dat $A(a, \bar{a})$.

Pentru aceasta se consideră planul ajutător Q dus prin preapta $D \equiv (AS)$ și care este proiectant (perpendicular) pe planul Π al reprezentării. Acest

¹⁾ Prezentată la sesiunea științifică a Institutului politehnic din Iași, din 28—30 septembrie 1967.

plan intersectează planul de proiecție (tabloul) după dreapta $U_0M \equiv (\Delta)$, care la rândul său se intersectează cu dreapta D în punctul $A_1(a_1, \bar{a}_1)$, care este punctul căutat.

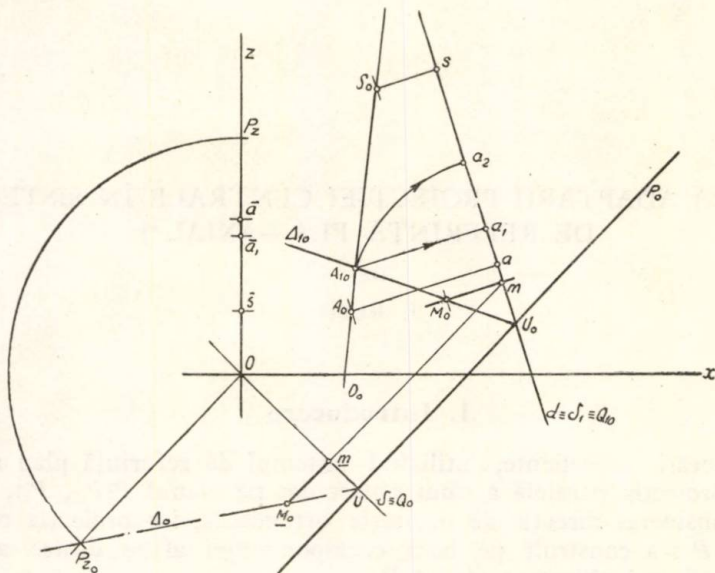


Fig. 1

Pentru găsirea acestui punct s-a rabătut în planul reprezentării Π planul proiectant Q împreună cu dreptele conținute, astfel încât D_0 și Δ_{10} determină prin intersecția punctul A_{10} , care ridicat împreună cu planul Q conduce la determinarea perspectivei $A_1(a_1, \bar{a}_1)$. Pentru obținerea pozițiilor rabătute (D_0) și (Δ_{10}) s-au utilizat punctele S și A , ce determină dreapta D și respectiv punctele U_0 și M , care determină dreapta Δ_1 . Astfel, din proiecțiile de pe planul reprezentării s , a și m s-au dus perpendiculare pe $d \equiv Q_{10} \equiv \delta_1$, pe care s-au măsurat cotele corespunzătoare.

Deci: $sS_0 = \bar{O}_s$, $aA_0 = O\bar{a}$ și $mM_0 = m\bar{M}$.

De observat este faptul că, cota punctului M considerat pe dreapta Δ_1 , deci în planul P , s-a găsit utilizând orizontala planului P ce trece prin acest punct. Planul proiectant al acestei orizontale intersectează planul proiectant al urmei axiale după o verticală și segmentul mM_0 de pe aceasta reprezintă cota punctului M . Prin urmare: $mM_0 = Z_M = m\bar{M}$.

3. Construcția perspectivei figurilor plane

3.1. Pătratul și cercul înscris, conținute într-un plan vertical

Fie pătratul $MABN$ și cercul înscris situat în planul vertical $R(R_0, R_{z\infty})$, precum și planul de front $P(P_0, P_{z\infty})$ pe care se obține perspectiva sau proiecția centrală a figurilor date. Ambele plane fiind proiectante (perpendiculare) pe planul Π al reprezentării, sînt paralele cu axa reprezentării O_x și de aceea urmele lor axiale vor fi puncte improprii, fiind notate cu $R_{z\infty}$ și respectiv $P_{z\infty}$.

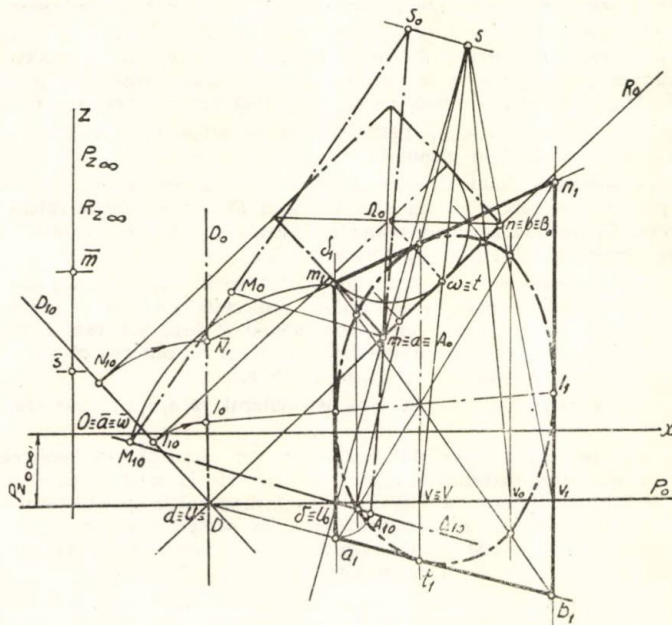


Fig. 2

Pentru determinarea omologiei (perspectivei) este necesar să fie dată o pereche de puncte omoloage (un punct de pe figura din planul Π și perspectiva sa de pe planul P).

De aceea se consideră planul vizual Q determinat de centrul de proiecție $S(s_1, s)$ și de latura AM verticală a pătratului considerat, plan care conține razele vizuale $SA(sa, sa)$ și $SM(sm, sm)$, fig. 2. Acest plan vertical se intersectează cu planul P după verticala $\Delta_1(\delta_1, \Delta_{10})$. Pentru obținerea punctelor de intersecție dintre razele SA și SM cu dreapta Δ_1 , în care aceste raze vizuale intersectează tabloul, se utilizează rabaterea. Astfel, atât proiectantele SA și SM cât și dreapta Δ_1 se rabatează în planul reprezentării Π , rabătînd pentru aceasta punctele care determină dreptele menționate. Se observă că proiecția δ_1 a verticalei de intersecție Δ_1 se găsește la intersecția dintre urmele Q_0 și P_0 , ea fiind un punct. Utilizîndu-se ca axă de rabatere urma Q_0 , dreapta Δ_1 se rabatează dacă prin $\delta_1 \equiv U_0 \equiv V_1$ se duce o perpendiculară la $Q_0 \equiv sa \equiv sm$.

Pentru rabaterea razelor vizuale SA și SM , se duce din $m \equiv A_0$ o perpendiculară tot pe Q_0 a cărei lungime este $mM_0 = Om$. Apoi din s se construiește de asemenea o perpendiculară la Q_0 pe care se măsoară cota acestui punct, ca rază de rabatere, astfel că $sS_0 = Os = Zs$.

Din rabaterea pe p'anul II a elementelor menționate, rezultă:

$$A_{10} = (S_0 A_0) \cap (\Delta_{10}) \quad \text{și} \quad M_{10} = (S_0 M_0) \cap (\Delta_{10}),$$

care sînt perspectivele punctelor A și B în rabatere, deci intersecțiile razelor vizuale corespunzătoare cu tabloul P .

Întrucît dreapta $\Delta_1(\delta_1, \Delta_{10})$ aparține și planului de front P , ea poate fi rabătută în planul II și în jurul urmei P_0 , obținându-se poziția $\delta_1 \perp P_0$, care este locul perspectivei laturii $AM(am, am)$ a pătratului dat. Deci perspectiva unei verticale pe un plan vertical este o verticală și ea este dreapta de intersecție dintre planul vizual vertical și tab'ou ca plan vertical. Pentru găsirea perspectivei $a_1 m_1$ se duc arcele de cerc concentrice în $U_0 \equiv \delta_1$ și de raze $U_0 A_{10}$ și $U_0 M_{10}$, care la intersecția cu δ_1 determină perspectivele menționate a_1 și m_1 .

Deci segmentul de dreaptă $a_1 m_1$ perpendicular pe urma P_0 și deci și pe axa Ox este perspectiva laturii AM , rabătută în planul II.

Axa de perspectivitate fiind dreapta de intersecție dintre planul R al figurilor geometrice date și planul P al tabloului, ea este verticala $D(d_1 D_{10})$ care se rabate în planul II întîi în jurul urmei R_0 (ducînd perpendiculara D_{10} la urma R_0), iar apoi în jurul urmei P_0 , ca perpendiculară la aceasta în $d \equiv U \equiv D$, notată cu D_0 .

Pe D_{10} se proiectează latura pătratului, astfel încît $nN_0 = UN_{10}$ sau $nN = UN_0$. Segmentul UN_0 este latura verticală a pătratului, proiectată pe axa de perspectivitate D_0 .

Planul vizual $[SB, SN]$ se intersectează cu tabloul P după verticala $B_1 N_1$, care rabătută în planul II în jurul urmei P_0 este perspectiva $b_1 n_1$. Se observă că perspectivele b_1 și n_1 rezultă din intersecția dreptelor Ua_1 și $N_0 m_1$ cu $(b_1 n_1)$.

Deci, perspectiva pătratului $MABN$ este patrulaterul $m_1 a_1 b_1 n_1$, care are laturile $a_1 m_1$ și $b_1 n_1$ verticale.

Pentru trasarea perspectivei cercului înscris, se vor construi perspectivele diametrilor orizontal și vertical ale căror extremități reprezintă punctele în care cercul înscris este tangent la laturile pătratului, notate cu I și T . Ducînd diagonalele patrulaterului perspectivă, se obține perspectiva centrului Ω , adică punctul ω_1 , iar perspectiva diametrului vertical este $\omega_1 t_1$, care trece prin $v \equiv V$. Acest punct rezultă din intersecția razei ST cu tabloul P , în epură din intersecția st cu urma P_0 . Deci verticala $(VT_1) \in [ST, S\Omega]$.

Dacă acum se consideră un punct I situat pe laturile verticale și la jumătatea lungimii acestora, atunci el se proiectează pe axa de perspectivitate (se prelungește diametrul orizontal (ΩI) , obținându-se I_0 pe D_{10} și apoi I_0 pe D_0). Dreapta $I_0 \omega_1$ determină pe perspectivele laturilor verticale punctele I_1 care împreună cu t_1 reprezintă punctele în care elipsa ca perspectivă a cercului înscris este tangentă la laturile patrulaterului ca perspectivă a pătratului considerat.

Pentru găsirea altor puncte ale elipsei înscrise, se consideră razele vizuale prin punctele în care cercul este intersectat de diagonalele pătratului înscris. Planele vizuale verticale ale acestora intersectează planul P după verticalele ce trec prin punctele $v_0 = V_0$. La rîndul lor, aceste verticale rabătute în jurul urmei P_0 intersectează perspectivele diagonalelor în punctele căutate. În acest mod poate fi trasată elipsa ω_1 , ca perspectivă a cercului Ω înscris în pătratul $MABN$.

4. Proiecția centrală a corpurilor geometrice

Pentru construcția perspectivei corpurilor geometrice s-a considerat că acestea au una din fețe situate în planul II al reprezentării și că sînt corpuri drepte.

4.1. Paralelipipedul dreptunghic

Ca exemplu s-a reprezentat în sistemul de referință plan-axial paralelipipedul dreptunghic $ABCEMNKL$ cu fața $ABCE$ situată în planul Π și centrul de proiecție $S(s, \bar{s})$, fig. 3. Fețele sale laterale sînt perpendiculare pe planul reprezentării, deci verticale.

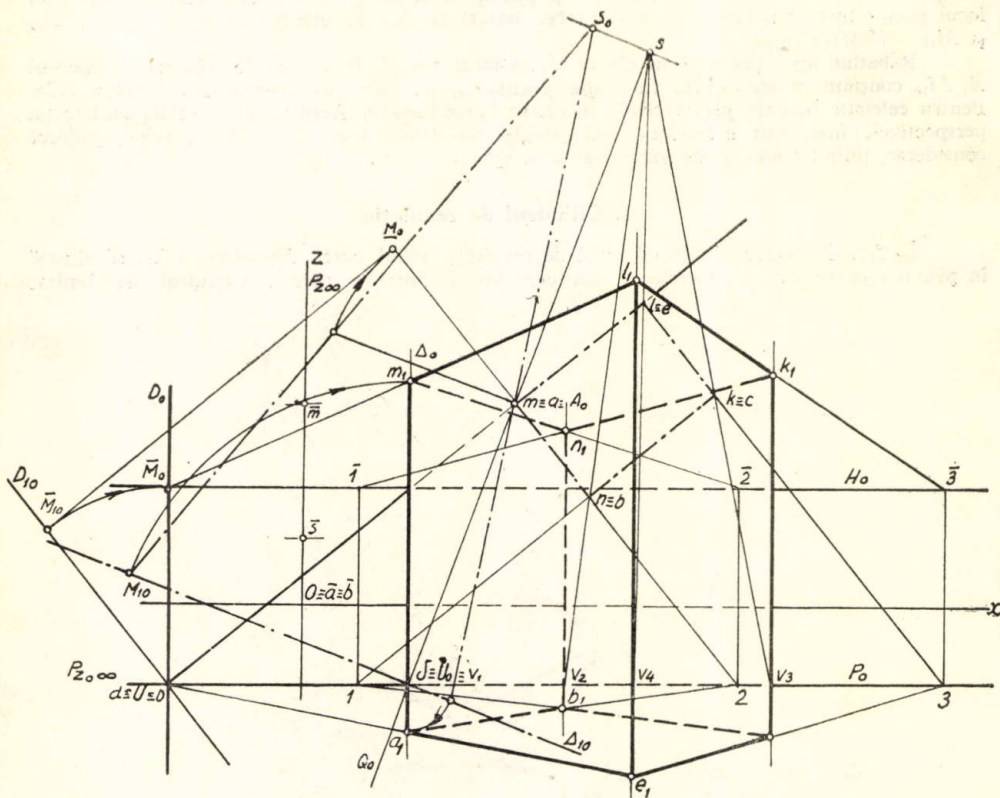


Fig. 3

Aceasta denotă că și muchiile laterale ale paralelipipedului sînt verticale, care se proiectează fiecare pe planul reprezentării după un punct. Conul vizual va fi format din razele vizuale care au ca punct comun centrul de proiecție, fiecare trecînd prin cîte un vîrf al poliedrului considerat.

Și aici, ca și în cazul precedent, se observă că fiecare pereche de raze vizuale ce trec prin cele două vîrfuri de pe aceeași muchie verticală determină planul vizual corespunzător, care este un plan vertical. Așa, spre exemplu, este planul Q , determinat de razele vizuale $SA(sa, \bar{s}a)$ și $SM(sm, \bar{s}m)$, punctele $A(a, \bar{a})$ și $M(m, \bar{m})$ fiind vîrfurile paralelipipedului, aparținînd aceleiași muchii verticale AM .

Se consideră acest plan rabătut pe planul Π al reprezentării împreună cu elementele conținute. Pentru aceasta, așa cum s-a arătat și în alte cazuri, se utilizează cotele punctelor ce determină razele vizuale din planul Q . Astfel, se duc perpendiculare la Q_0 (axa de raba-

tere) prin proiecțiile S , m , a și prin $\delta \equiv U_0 \equiv V_1$. Pe aceste perpendiculare se măsoară cotele corespunzătoare, care sînt:

$$sS_0 = \bar{O}s = Z_s, aA_0 = \bar{O}a = Z_A = 0 \text{ și } mM_0 = \bar{O}m = Z_M.$$

Planul de proiecție $P(P_0, P_{\infty})$ fiind un plan de front, deci un plan vertical, rezultă că dreapta de intersecție dintre acest plan și planul ajutător Q este o verticală, ce reprezintă locul perspectivei muchiei AM în rabatere, notată cu Δ_0 . Se observă că $A_{10} = (SA) \cap (\Delta_{10})$ și $M_{10} = (SM) \cap (\Delta_{10})$.

Rabatind apoi planul P în planul Π , în jurul urmei P_0 ca axă de rabatere, segmentul $A_{10}M_{10}$ conținut în acest plan va ocupa poziția a_1m_1 , care este perspectiva muchiei AM . Pentru celelalte muchii, perspectivele se găsesc în mod analog. Aceasta face posibilă construcția perspectivei, mai întâi a fețelor (vezi perspectiva pătratului), iar apoi a paralelipipedului considerat, ținînd seama și de vizibilitatea în epură.

4.2. Cilindrul de revoluție

În fig. 4 s-a considerat cilindrul de revoluție avînd curba directoare $\Omega(\omega, \bar{\omega})$ situată în planul reprezentării Π . Cercul se consideră înscris într-un pătrat. Cilindrul este limitat

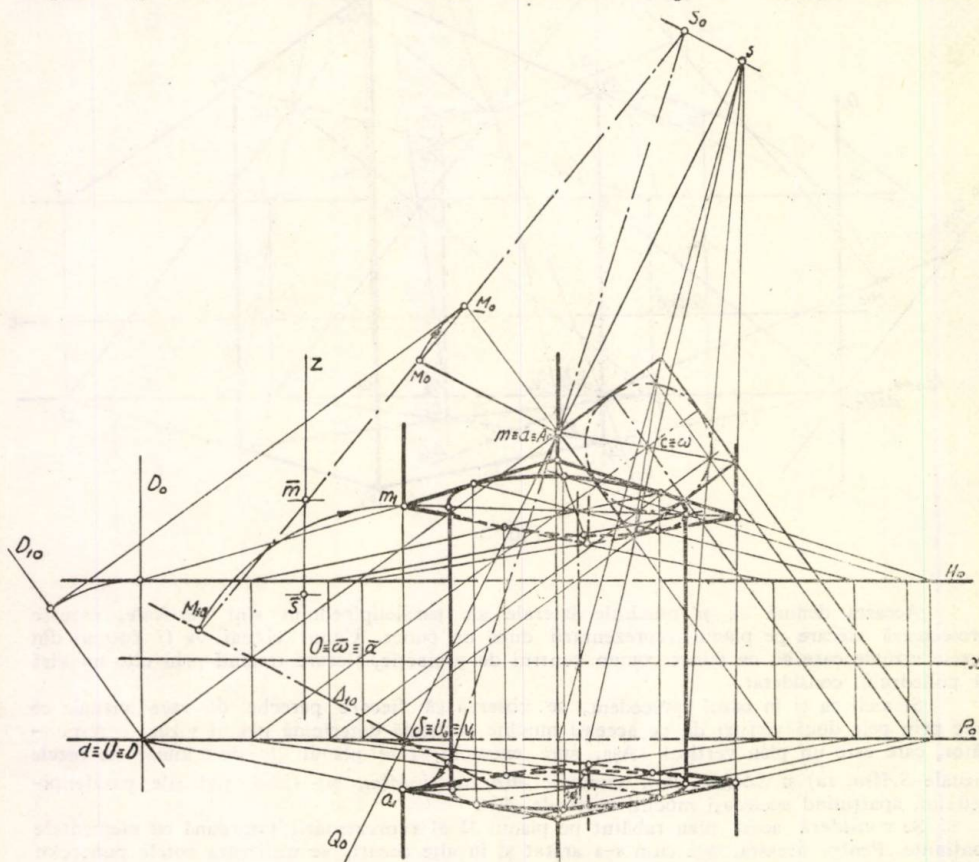


Fig. 4

de planul de nivel H de cotă $Z_H = mM_0 = ,...$, punctul $C(c, \bar{c})$ fiind centrul cercului secțiunii drepte în cilindru considerat cu planul H , prin urmare de aceeași rază $R_\Omega = R_C$, înscris într-un pătrat egal cu cel din planul Π și cu laturile paralele cu acesta. Ca și în cazul precedent, s-a considerat conul vizual ale cărui proiectante sînt determinate de centrul de proiecție $S(s, \bar{s})$ și de punctele caracteristice ale corpului geometric considerat.

Se menționează, în acest sens, centrele cercurilor Ω și C , vîrfurile pătratelor circumscrise, extremitățile diametrilor principali în cele două cercuri, precum și punctele în care diagonalele acestora intersectează cercurile menționate.

Construcția perspectivei acestui element de arhitectură reprezintă o sinteză a principiilor indicate în lucrare.

5. Concluzii

Pentru construcția în epură a imaginii obiectului (perspectivei) s-a considerat sistemul de referință plan-axial, cu ajutorul căruia s-au reprezentat pe planul Π și pe axa O_z conul vizual, prin centrul lui de proiecție și obiectul dat.

Intersecția unor raze vizuale cu tabloul s-a făcut pentru determinarea omologiei (perspectivei) dintre figurile din planele Π și H și proiecțiile centrale ale acestora pe tabloul P , iar construcția perspectivei s-a realizat prin utilizarea corespondenței perspective dintre figurile geometrice din planele menționate ca proiecții ale obiectului dat, și proiecțiile centrale ale acestora pe tablou.

Lucrarea este utilă în proiectare pentru elaborarea unor piese din partea desenată a proiectelor din domeniul construcțiilor și arhitecturii, precum și din alte domenii ale tehnicii.

Primită la 28 august 1970

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EINE UMARBEITUNG DER ZENTRALEN PROJEKTION FÜR AXIALEBENEM REFERENZ-SYSTEM

(Zusammenfassung)

In vorliegender Arbeit wurde die Projektions-Richtung, durch die Projektions-Mitte $S(s, S_0)$ ersetzt.

Die Perspektive eines gegebenen Körpers wurde in diesem Falle mittels Durchschnitt mit Ebene $P(P_0, P_0)$ des Sichtkegels, welcher durch die in S zusammenwirkenden, die die charakteristischen Punkte des Körpers überschneidenden Sichtstrahlen, bestimmt ist.

In diesem Falle wurde die Omologie-Korrespondenz zwischen der orthogonalen Projektion des Körpers in der Ebene Π der Darstellung, und die Perspektive welche in Bild P konstruiert werden muss, verwendet.

SOME QUESTIONS IN THE CLASSICAL THEORY OF VIBRATING SYSTEMS

BY

PETER LANCASTER

1. Preliminaries

Let A, C be hermitian (or real symmetric) $n \times n$ matrices with A positive definite and C nonnegative definite¹⁾. It is very well known that equations of the form $(A\mu - C)\mathbf{q} = \mathbf{0}$ for the scalar μ and the column vector $\mathbf{q}$ arise in a variety of applications. We suppose that this equation refers to some „system“ which will be unspecified. (It may be mechanical or electrical, for example.) It is also very well known that there are at most n distinct values of μ for which non-zero solution vectors $\mathbf{q}$ exist, and that these latent roots, μ , are all non-negative (and positive if C is positive definite). We therefore write $\mu_1 = \omega_1^2, \dots, \mu_n = \omega_n^2$ for these preferred values of μ , and $\omega_1, \dots, \omega_n$ are known as the *natural frequencies* of vibration of the system. Note that the n natural frequencies are not necessarily distinct. For each ω_j there exists a $\mathbf{q}_j \neq \mathbf{0}$ such that $(A\omega_j^2 - C)\mathbf{q}_j = \mathbf{0}$, ($j = 1, 2, \dots, n$) and these vectors describe the *modes of vibration* corresponding to the natural frequencies. We denote transposed vectors and matrices with a „ $*$ “ and transposed conjugates a „ $*$ “. It may be assumed [2] that the vectors $\mathbf{q}_j$ are normalized in such a way that the following orthogonality conditions are satisfied $\mathbf{q}_j^* A \mathbf{q}_k = \delta_{jk}$, $\mathbf{q}_j^* C \mathbf{q}_k = \omega_j^2 \delta_{jk}$ for ($j, k = 1, 2, \dots, n$). Writing Q for the square matrix whose columns are $\mathbf{q}_1, \dots, \mathbf{q}_n$, respectively, I for the unit matrix, and $W = \text{diag} \{ \omega_1^2, \dots, \omega_n^2 \}$, these conditions may be written

$$(1) \quad Q^* A Q = I, \quad Q^* C Q = W.$$

In algebraic terminology this corresponds to the result that the matrices A and C can be simultaneously reduced to diagonal matrices by a congruent transformation (that defined by Q , for example).

2. Sensitivities of the Natural Frequencies

In reference [5] there is some discussion of the effect of perturbations in A and C on the natural frequencies. As the authors point out this is not a new question and the classical analysis of Rayleigh remains

¹⁾ We follow the terminology of [2].

the standard reference work on the subject. However, Soong and Bogdanoff pay special attention to the phenomenon that the lower „natural frequencies are relatively stable“ under perturbations of A and the higher natural frequencies „are much less so“. However, if one observes the *relative* changes in the natural frequencies (and this is easily done in the example of [5]) this effect disappears and all changes seem to be of the same order.

We wish to define a quantitative measure for the sensitivities of the natural frequencies to perturbations which will reflect the above effects. A general perturbation will, of course, disturb both the frequencies and modes and in order to define our problem more precisely we choose only those perturbations which will leave the modes invariant. We shall see that for unrepeatd, non-zero natural frequencies this leads to a unique value of the sensitivity.

Consider first perturbations in A only giving rise to a new coefficient matrix $\hat{A}$. We seek a system for which the new frequencies will be $\omega_j^2(1 + \delta_j)$, ($j = 1, 2, \dots, n$). Thus we do not permit variations in the zero „frequency“, should it appear. Let $\Delta = \text{diag}\{\delta_1, \dots, \delta_n\}$ and consider the hermitian matrix $\hat{A} = A - (Q^*)^{-1} \Delta (I + \Delta)^{-1} Q^{-1}$.

Then using (1), $Q^* \hat{A} Q = I - \Delta (I + \Delta)^{-1} = (I + \Delta)^{-1}$. Thus the system corresponding to $\hat{A} \mu - C$ has latent roots $\omega_j^2(1 + \delta_j)$ and the same modes of vibration as $A \mu - C$. Now suppose that we perturb only the k^{th} frequency.

Thus, we set $\delta_j = 0, j \neq k$, and in this case write $\Delta = \Delta_k, \hat{A} = \hat{A}_k$. We wish to relate the magnitude of δ_k to the magnitude of the perturbation producing the change in frequency. Note that from (1) we may write $Q^{-1} = Q^* A$, and we have $\hat{A}_k - A = -A Q \Delta_k (I + \Delta_k)^{-1} Q^* A$.

Observe that the j^{th} column of AQ is Aq_j and the j^{th} row of $Q^* A = (AQ)^*$ is $(Aq_j)^*$ and it follows ([2] p. 20) that

$$\hat{A}_k - A = - \sum_j \frac{\delta_j}{1 + \delta_j} (Aq_j) (Aq_j)^* = - \frac{\delta_k}{1 + \delta_k} (Aq_k) (Aq_k)^*.$$

If $\|\cdot\|$ applied to a vector denotes its euclidean (or Frobenius) norm and applied to a matrix denotes either its spectral or euclidean norm ([2], ch. 6) then it is easily seen that

$$\|\hat{A}_k - A\| = \frac{\delta_k}{1 + \delta_k} \|Aq_k\|^2.$$

Thus, for small changes in the frequency the magnitude of the perturbation in A required to change ω_k^2 to $\omega_k^2(1 + \delta_k)$ is simply proportional to $\|Aq_k\|^2$, and this sensitivity can be measured given only q_k , with no knowledge of the other modes. We note in particular that this sensi-

tivity does not depend explicitly on ω_k^2 itself. Conversely, for a fixed magnitude of perturbation of A , and supposing $\|Aq_k\|$ of the same order for each k , we can produce changes in δ_k of similar magnitudes for each k . This means, of course, changes in the absolute values of the ω_k^2 proportional to their magnitudes, as observed in [5].

We apply a similar argument to perturbations in C . Let $\hat{C} = C + (Q^*)^{-1} \cdot W^2 \Delta Q^{-1}$. Then $Q^* \hat{C} Q = W^2 (I + \Delta)$ and it follows that $A\mu - \hat{C}$ has natural frequencies $\omega_j^2(1 + \delta_j)$, ($j = 1, 2, \dots, n$). Then write

$$\hat{C}_k - C = (AQ) W^2 \Delta_k (AQ)^* = \sum_j \delta_j \omega_j^2 (Aq_j) (Aq_j)^* = \delta_k \omega_k^2 (Aq_k) (Aq_k)^*$$

and we have $\|\hat{C}_k - C\| = |\delta_k| \omega_k^2 \|Aq_k\|^2$.

Thus, for a perturbation of C of fixed magnitude we can obtain magnitudes for δ_k inversely proportional to ω_k^2 , and absolute changes in ω_k^2 of the same magnitude for each k .

It is not difficult to see that $\hat{A}_k, \hat{C}_k$ are uniquely defined for an unrepeatd natural frequency, but that otherwise there is some ambiguity in the definition of q_k and hence of $\hat{A}_k, \hat{C}_k$.

To summarize: For perturbations in which modes of vibration are invariant and zero frequencies are unperturbed, the change in relative magnitude of the square of the frequency per unit perturbation may be measured by $\omega_k^{-2} \|Aq_k\|^{-2}$ for perturbations in C , and $\|Aq_k\|^{-2}$ for sufficiently small perturbations in A .

3. Perturbation Theory for Damped Systems

We now turn our attention to the more general question of vibrations of a system giving rise to the equations

$$(2) \quad (A\lambda^2 + B\lambda + C) \mathbf{x} = 0,$$

where A, C are as before and B is hermitian. A fairly general formulation of the perturbation problem is as follows: Let A_1, B_1, C_1 be hermitian matrices of order n and ζ a real parameter. For each value of ζ we have a latent root problem

$$(3) \quad [(A + \zeta A_1) \lambda^2 + (B + \zeta B_1) \lambda + C + \zeta C_1] \mathbf{x} = 0,$$

which reduces to (2) in case $\zeta = 0$. More generally, we can view the latent roots defined by (3) as functions of ζ and investigate the behaviour of the functions $\lambda(\zeta)$ in a neighbourhood of $\zeta = 0$. If

$p(\lambda, \zeta) = \det \{(A + \zeta A_1) \lambda^2 + (B + \zeta B_1) \lambda + (C + \zeta C_1)\}$,
then we have

$$(4) \quad p(\lambda(\zeta), \zeta) = 0.$$

The algebraic functions $\lambda(\zeta)$ are known to have expansions in fractional powers of ζ (known as Puiseux series) valid in a neighbourhood of $\zeta=0$ ([2], ch. 7), and if $\lambda(0)$ is an unpeated latent root of (2) then $\lambda(\zeta)$ is, indeed, analytic in a neighbourhood of $\zeta=0$. That this is not generally true is illustrated by the following example. Take

$$(5) \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3^{1/2}/2 \\ 3^{1/2}/2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

(all of which are positive definite) and the unperturbed eigenvalues are $(-3 \pm i(23)^{1/2})/4$, each taken twice. With $A_1 = C_1 = 0$ and

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

the latent roots which take values $\lambda_0 = (-3 + i(23)^{1/2})/4$ at $\zeta=0$ are given by $\lambda(\zeta) = \lambda_0 + a_1 \zeta^{1/2} + a_2 \zeta + \dots$, where a_1 is either of the square roots of $(31 - i5(23)^{1/2})/92$.

A much more simple case of non-analytic behaviour is, of course, obtained in the case $n=1$ with $p(\lambda, \zeta) = \lambda^2 + \zeta$. More generally, it is easily seen that in the case $n=1$ the equation

$$(a + \zeta a_1) \lambda^2 + (b + \zeta b_1) \lambda + c + \zeta c_1 = 0$$

has solutions $\lambda(\zeta)$ analytic in a neighbourhood of $\zeta=0$ if (i) $b^2 \neq 4ac$, or (ii) $b^2 = 4ac$ and $b_1 b \neq 2(a_1 c - a c_1)$.

Conversely, if $b^2 = 4ac$ and $b_1 b \neq 2(a_1 c - a c_1)$ then the solutions are not analytic functions of ζ in any neighbourhood of $\zeta=0$.

Notice that the case $b^2 = 4ac$ is often known as that of *critical damping* and it is in this case only that (for $n=1$) we generally have non-analytic behaviour of the functions $\lambda(\zeta)$. The example (5) shows that for $n > 1$ non-analytic behaviour is no longer confined to cases of unperturbed latent roots which are critically damped.

An important case in which analytic behaviour of the perturbed latent roots is guaranteed is that in which all the unperturbed latent roots are known to be *real*. Since the latent roots of $A\lambda^2 + B\lambda + C$ are also eigenvalues of the matrix

$$M = \begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix}$$

and conversely, the fact that all the perturbed latent roots $\lambda(\zeta)$ are analytic in a neighbourhood of $\zeta=0$ then follows from a theorem of Porsching [4]. In particular, real latent roots and hence analytic perturbations are guaranteed in the case of *over-damping* [1, ch. 7], which is defined as

follows: Define the real-valued functional a on $\mathcal{C}_n$ by $a(\mathbf{x}) = \mathbf{x}^* A \mathbf{x}$, and define functionals b and c similarly in terms of B and C . The system (2) is over-damped if, for all non-zero $\mathbf{x} \in \mathcal{C}_n$, $b(\mathbf{x})^2 > 4a(\mathbf{x})c(\mathbf{x})$.

It should be remarked that although this appears to be a condition which merely prevents cases of critical damping, it does much more than that. It is vital for the application of Porsching's results that *all* the latent roots be real.

4. Differentiable Latent Roots

If a Puiseux series

$$\lambda(\zeta) = a_0 + a_1 \zeta^{1/m} + a_2 \zeta^{2/m} + \dots$$

has $a_1 = a_2 = \dots = a_{m-1} = 0$ we shall say that $\lambda(\zeta)$ is differentiable at $\zeta = 0$. Clearly, the first non-zero coefficient in the sequence $a_1, a_2, \dots$ is most important in determining the rate of growth of λ with ζ . In particular, if we can prove that $\lambda(\zeta)$ is merely differentiable at $\zeta = 0$ (and not necessarily analytic), this may yet be a valuable piece of information.

Now the latent roots of $A\lambda^2 + B\lambda + C$ not only coincide with the eigenvalues of the matrix M , but the array of elementary divisors is also the same in both cases (this follows from the preservation of the Jordan chains, [3]). There is then a theorem (Theorem 7.9.1. of [2]) that the eigenvalue $\lambda(\zeta)$ of $M(\zeta)$ with $\lambda(0) = \lambda_0$ is differentiable at $\zeta = 0$ provided λ_0 has only linear elementary divisors. Thus, the perturbed latent root $\lambda(\zeta)$ for which $\lambda(0) = \lambda_0$ is differentiable at $\zeta = 0$ if the latent root λ_0 of $A\lambda^2 + B\lambda + C$ has only linear elementary divisors.

The nature of the elementary divisors is generally difficult to ascertain directly, so we look for other conditions which will guarantee linear elementary divisors. Since it is often necessary to find latent vectors explicitly, the following characterization may be useful (Theorem 4.6 of [1]):

Lemma. Let λ_0 be a latent root of the regular λ -matrix $D(\lambda)$ and $D'(\lambda)$ be the λ -derivative of $D(\lambda)$. There exists a non-linear elementary divisor of D corresponding to λ_0 if there exists an $\mathbf{x} \neq 0$ such that $D(\lambda_0)\mathbf{x} = 0$ and $\mathbf{y}'D'(\lambda_0)\mathbf{x} = 0$ for all $\mathbf{y}$ such that $\mathbf{y}'D(\lambda_0) = 0$.

We can use this result to obtain another rather interesting condition in an important special case. Suppose now that A, B, C , are real and symmetric. Define the functional $\alpha(\mathbf{x})$ on $\mathcal{C}_n$ by $\alpha(\mathbf{x}) = \mathbf{x}' A \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_n$ and define functionals β, γ similarly in terms of B and C . Note that in contrast to functionals a, b, c , defined above, α, β , and γ are complex-valued.

Theorem. Let A, B, C , be real symmetric matrices with $\det A \neq 0$ and let λ_0 be a latent root of $A\lambda^2 + B\lambda + C$. If for every $\mathbf{x} \neq 0$ such that $(A\lambda_0^2 + B\lambda_0 + C)\mathbf{x} = 0$ we have $\beta(\mathbf{x})^2 \neq 4\alpha(\mathbf{x})\gamma(\mathbf{x})$, then λ_0 has only linear elementary divisors.

Proof. We prove the contrapositive statement: If λ_0 has a non-linear elementary divisor, then there exists an $\mathbf{x} \neq 0$ such that $\beta(\mathbf{x})^2 = 4\alpha(\mathbf{x})\gamma(\mathbf{x})$.

Thus, given that λ_0 has a non-linear divisor it follows from the lemma that there is an $\mathbf{x} \neq 0$ such that $(A\lambda_0^2 + B\lambda_0 + C)\mathbf{x} = 0$ and $\mathbf{y}'(2A\lambda_0 + B)\mathbf{x} = 0$ for all $\mathbf{y}$ such that $\mathbf{y}'(A\lambda_0^2 + B\lambda_0 + C) = 0'$. We first observe that $A\lambda_0^2 + B\lambda_0 + C$ is a symmetric matrix (possibly complex symmetric, but not complex hermitian) and so we may choose $\mathbf{y} = \mathbf{x}$. Thus we have $2\alpha(\mathbf{x})\lambda_0 + \beta(\mathbf{x}) = 0$, and we also have $\alpha(\mathbf{x})\lambda_0^2 + \beta(\mathbf{x})\lambda_0 + \gamma(\mathbf{x}) = 0$. It follows at once that $\beta(\mathbf{x})^2 = 4\alpha(\mathbf{x})\gamma(\mathbf{x})$, as required.

A similar argument can be used to prove that latent roots of over-damped systema have only linear elementary divisors (cf. [1, Theorem 7.3]).

In example (5) the latent root $\lambda_0 = (-3 + i(23)^{1/2})/4$ has a corresponding latent vector

$$\mathbf{x} = \begin{bmatrix} 3^{1/2}(1 + i(23)^{1/2}) \\ 6 \end{bmatrix},$$

and it is found that $\beta(\mathbf{x})^2 = 4\alpha(\mathbf{x})\gamma(\mathbf{x})$ so we have no reason to expect differentiable perturbed latent roots with $\lambda(0) = \lambda_0$, as indeed they are not. It is also found that $b(\mathbf{x})^2 \neq 4a(\mathbf{x})c(\mathbf{x})$.

It will be seen that throughout the search for conditions for analytic or differentiable perturbed latent roots the functionals $b(\mathbf{x})^2 - 4a(\mathbf{x})c(\mathbf{x})$ and $\alpha(\mathbf{x}) - 4\beta(\mathbf{x})\gamma(\mathbf{x})$ seem to play vary tantalising roles whose significance is still not completely understood.

5. Perturbations about Diagonable Systems

We next devote our attention to equations of the form given in (2) which are „close“ to another equation with a much simpler algebraic structure. The simplest problems of this type which are not entirely trivial are those for which A , B , and C can be simultaneously reduced to diagonal form by a congruent transformation (cf. remarks on p. 129 of [1]). In this case we obtain scalar quadratic equations for λ . Our next theorem suggests that the class of such problems is very small, but the class of problems which can be analyzed by considering perturbations (as in (3)) from diagonable systems is more significant.

Theorem. *Let A , B , C be hermitian matrices of order n and A be positive definite. Then A , B , and C can be reduced to diagonal matrices by one and the same congruent transformation if $A^{-1}B$ and $A^{-1}C$ commute.*

Proof. Suppose there exists a Q with $\det Q \neq 0$ and $Q^*AQ = D_1$, $Q^*BQ = D_2$, $Q^*CQ = D_3$, where D_1, D_2 , and D_3 are diagonal matrices. Then $\det D_1 \neq 0$ and we may write $Q^* = D_1Q^{-1}A^{-1}$ in the last two equations to obtain $Q^{-1}(A^{-1}B)Q = D_1^{-1}D_2$, $Q^{-1}(A^{-1}C)Q = D_1^{-1}D_3$.

Solving for $A^{-1}B$ and $A^{-1}C$ we verify at once that they commute. Note that in this part of the proof we need only assume that A is non-singular and not necessarily positive definite.

Conversely, suppose it given that $A^{-1}B$ and $A^{-1}C$ commute. From the theorem concerning the simultaneous reduction of two matrices we obtain the existence of a Q such that $Q^*AQ = I$, $Q^*BQ = D_2$, where $D_2 = \text{diag} \{d_1, \dots, d_1, d_2, \dots, d_s\}$, $d_i \neq d_j$ when $i \neq j$ and d_j appears α_j times. It follows, as above, that $Q^{-1}(A^{-1}B)Q = D_2$ and hence $A^{-1}B = Q D_2 Q^{-1}$. Since $(A^{-1}B)(A^{-1}C) = (A^{-1}C)(A^{-1}B)$ we now have $D_2 Q^{-1}(A^{-1}C)Q = Q^{-1}(A^{-1}C)Q D_2$, or $D_2(Q^*CQ) = (Q^*CQ)D_2$, which implies

$$(6) \quad Q^*CQ = \text{diag} \{\Delta_1, \dots, \Delta_s\},$$

where Δ_j is an arbitrary hermitian matrix of order α_j .

Using Theorem 2.2 of [1] we may suppose that Q has the form $Q = [Q_1 G_1 Q_2 G_2 \dots Q_s G_s]$, where Q_j is $n \times \alpha_j$, and G_j is an arbitrary unitary matrix of order α_j , ($j = 1, 2, \dots, s$), and substituting in (6) we now have $G_j^* Q_j^* C Q_j G_j = \Delta_j$, or $Q^* C Q = G_j \Delta_j G_j^*$, ($j = 1, 2, \dots, s$).

Now, for each j , choose G_j to be a unitary matrix reducing Δ_j to diagonal form and we thus reduce Q^*CQ to diagonal form. This completes the proof.

To illustrate how restrictive the diagonal case is we observe that $A^{-1}B$ and $A^{-1}C$ commute if one of these matrices is a (scalar) polynomial in the other.

In the general (non-diagonal) case with $A = I$ we can produce a diagonalable trio I, B_0, C by choosing for B_0 the „best“ approximation of B by a polynomial p in C . Thus, we take

$$B_0 = p(C) = a_0 I + a_1 C + \dots + a_{k-1} C^{k-1},$$

where k is the degree of the minimal polynomial of C and $a_0, \dots, a_{k-1}$ are chosen to minimize $\|B - B_0\|$. The determination of the coefficients is a routine process if the euclidean matrix norm is used. The matrix B is then taken as a perturbation of B_0 . However, this does not seem to be a process which can be recommended for general use, because B_0 may still not be close enough to B for perturbation techniques to be valid.

6. Perturbation Coefficients

We now consider the diagonal case and suppose the reduction to diagonal matrices to have been performed so that the unperturbed problem (3) is given by $A = I$,

$$B = \text{diag} \{b_1, \dots, b_n\}, \quad C = \text{diag} \{\omega_1^2, \dots, \omega_n^2\},$$

(we again assume C to be positive definite). The $2n$ unperturbed latent roots are therefore given by

$$(7) \quad 2\lambda_j(0) = -b_j \pm i(4\omega_j^2 - b_j^2)^{1/2}, \quad (j = 1, 2, \dots, n).$$

It is clear that in this case the latent root $\lambda_j(0)$ has a non-linear elementary divisor if it is a double zero of $\lambda^2 + b_j\lambda + \omega_j^2$. That is, if $\lambda_j(0)$ corresponds to a critically damped mode. Thus, we may say that, *if no mode of the unperturbed system is critically damped, then all the perturbed latent roots are differentiable*. With the above matrices A , B , C , and denoting the elements of A_1 by a_{jk} etc., we can now apply standard perturbation techniques [1, ch 9] to find the leading coefficient a_{2j} in the expansion

$$\lambda_j(\zeta) = \lambda_j(0) + a_{2j}\zeta + a_{3j}(\zeta^{1/2})^3 + \dots$$

It is found that, if $\omega_j \neq 0$

$$(8) \quad a_{2j} = \pm \frac{i}{(4\omega_j^2 - b_j^2)^{1/2}} (\mu_j^2 a_{jj} + \mu_j b_{jj} + c_{jj}),$$

where the choice of sign accords with that for μ_j in (7). If $\omega_j = 0$, we have $a_{2j} = b_j a_{jj} - b_{jj}$, or $a_{2j} = 0$.

The formula (8) is first of interest because it shows immediately the effect of the magnitude of $(4\omega_j^2 - b_j^2)^{1/2}$ and the proximity to critical damping.

The perturbation problem considered previously in most detail is that of light damping of an otherwise conservative system [1, ch 9]. This can be viewed as a special case of our problem with $B = 0$ and $A_1 = C_1 = 0$. In this case critical damping arises only when $b_j = \omega_j^2 = 0$.

Received April 15, 1969

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UNELE PROBLEME PRIVIND TEORIA CLASICĂ A SISTEMELOR OSCILANTE

(Rezumat)

Utilizând o serie de rezultate bazate în parte pe cercetări proprii [1], [2], autorul studiază sensibilitatea frecvențelor naturale, perturbația coeficienților și alte probleme interesând teoria clasică a sistemelor oscilante.

MOUVEMENTS POLYTROPICI LE LONG D'UNE FAMILLE DE PLANS PARALLÈLES (I)

PAR

C. L. ACKER

1. Soit le repère local triorthogonal $T(M; I_1, I_2, I_3)$ ayant son origine M située à un point quelconque du domaine D occupé par le fluide, I_3 étant égal à $\bar{v}/v$ ($\bar{v}$ est le vecteur vitesse) et I_2 — normal aux plans de la famille P de plans parallèles, le long desquels se développe le mouvement.

Considérons de plus l'espace des 4 paramètres suivants: les coordonnées d'un point quelconque du domaine D appartenant à l'espace euclidéen E_3 et le temps t .

Si à un déplacement élémentaire dans l'espace des 4 paramètres précédents, la variation du premier ordre du repère T est donnée par

$$d\bar{M} = \omega^i I_i, \quad dI_i = \omega_j^i I_j = \bar{\omega} \times I_i, \quad (i, j = 1, 2, 3),$$

où

$$\omega = p I_1 + q I_2 + r I_3 \quad (p = \omega_2^3 = -\omega_3^2, \quad q = \omega_3^1 = -\omega_1^3, \quad r = \omega_1^2 = -\omega_2^1)$$

et

$$p = p_i \omega^i + p_t dt, \quad q = q_i \omega^i + q_t dt, \quad r = r_i \omega^i + r_t dt,$$

alors l'ensemble des conditions géométriques, cinématiques et dynamiques, sous lesquelles se développe le mouvement polytropique tridimensionnel instationnaire d'un fluide idéal le long d'une famille de plans parallèles, constitue le système différentiel extérieur suivant¹⁾:

¹⁾ On a utilisé de nouveau [1] les opérateurs „grad“, „div“, „ $\bar{v} \nabla$ “ exprimés à l'aide des produits extérieurs spéciaux de S. S. Bucheguence et du prof. Gh. Gheorghiev [2], pour obtenir le système (Z).

$$\begin{aligned}
 (1) \quad & p = f_i \omega^i + p dt, \quad q = q_i \omega^i + q_t dt, \quad r = r_i \omega^i + r_t dt, \\
 (2) \quad & d\omega^1 = -[\omega^2, r] + [\omega^3, q], \\
 (3) \quad & d\omega^2 = [\omega^1, r] - [\omega^3, p], \\
 (4) \quad & d\omega^3 = -[\omega^1, q] + [\omega^2, p], \\
 (5)...(7) \quad & dp = [r, q], \quad dq = [p, r], \quad dr = [q, p], \\
 (8) \quad & r_i, r_t, p_i, p_t = 0, \quad (i = 1, 2, 3), \\
 (9) \quad & \rho_t + v \rho_3 + \rho [v_3 + v(q_1 - p_2)] = 0, \\
 (10) \quad & v(q_t + v q_3) - \Phi_1 = -K \chi \rho^{\chi-2} \rho_1, \\
 (11) \quad & v(p_t + v p_3) + \Phi_2 = K \chi \rho^{\chi-2} \rho_2, \\
 (12) \quad & v_t + v v_3 - \Phi_3 = -K \chi \rho^{\chi-2} \rho_3, \\
 (13) \quad & dv = v_i \omega^i + v_t dt \\
 (14) \quad & d\rho = \rho_i \omega^i + \rho_t dt.
 \end{aligned}
 \tag{Z}$$

Les équations (1) représentent les relations entre les projections du vecteur Darboux correspondant au repère T et les formes Pfaff indépendantes ω^i, dt ($i = 1, 2, 3$); les équations (2) ... (7) sont les équations de la structure de l'espace euclidéen E_3 ; les équations (8) proviennent du caractère constant du vecteur unité I_2 , $dI_2 = -rI_1 + pI_3 = 0$; l'équation (9) est l'équation de la continuité; les équations (10) ... (12) proviennent

de l'équation d'Euler $\bar{a} = \bar{\Phi} - \frac{1}{\rho} \text{grad } b$, par l'élimination de la pression b à l'aide de l'équation de l'état, $b = K \rho^\chi$, des transformations polytropiques; enfin les équations (13), (14) sont des relations entre les fonctions inconnues qui interviennent dans les équations précédentes.

En considérant les mouvements tridimensionnels instationnaires, toutes les formes extérieures de (Z) et du système (Z) fermé appartiendront à l'anneau grassmannien $G(\omega^1, \omega^2, \omega^3, dt)$.

Les variétés intégrales de (Z) le long desquelles $[\omega^1, \omega^2, \omega^3, dt] \neq 0$ offrent des informations sur les transformations polytropiques tridimensionnelles instationnaires d'un fluide idéal le long d'une famille P de plans parallèles; notamment l'arbitraire de ces variétés est même celui des transformations considérées.

2. Pour préciser l'arbitraire de ces variétés intégrales on doit chercher l'involution de (Z), que nous avons établie sans faire appel aux prolongements du 2-ème ordre, pour le cas général de transformations considérées, aussi que pour 22 cas particuliers de ces transformations.

Les résultats obtenus à cet égard sont présentés ci-dessous.

1. La classe des transformations polytropiques tridimensionnelles instationnaires d'un fluide idéal le long d'une famille de plans parallèles dépend

de 3 fonctions arbitraires de 3 arguments lorsque les fonctions Φ_3, Φ_1 sont données; si $2-i$ fonctions de l'ensemble (Φ_3, Φ_1) sont prescrites, alors la classe précédente dépend de i fonctions arbitraires de 4 arguments. Pour ce cas général, le système (Z) fermé comprend 17 fonctions inconnues $(q_i, q_i, v, v_i, v_i, \rho, \rho_i, \rho_i, \Phi_i; i = 1, 2, 3)$ et les caractères réduits sont: $s_0 = 6, s_1 = s_2 = s_3 = 3$.

2. La classe particulière de la classe précédente, pour laquelle $\text{grad } v = 0$, c'est-à-dire pour laquelle les mouvements se développent tel que la grandeur de la vitesse est la même à tous les points de D , à un moment quelconque donné, dépend de 2 fonctions arbitraires de 3 arguments lorsque Φ_1 est donnée; si les forces massiques ne sont point données, alors cette classe dépend de 1 fonction arbitraire de 4 arguments (on a attaché au système (Z) les équations $v_i = 0, (i = 1, 2, 3)$, qui caractérisent cette classe dans la classe (1)).

3. La classe particulière de la classe (1), qui comprend les mouvements d'un fluide homogène à un moment quelconque donné, dépend de 2 fonctions arbitraires de 3 arguments lorsque Φ_1 est donnée; si les forces massiques ne sont pas prescrites alors cette classe dépend de 1 fonction arbitraire de 4 arguments (on a attaché les équations $\rho_i = 0, (i = 1, 2, 3)$, qui caractérisent cette classe dans la classe (1), au système (Z)).

4. La classe particulière de la classe (1), dont les mouvements se développent tel qu'à tous les points du domaine D occupé par le fluide, la direction de la vitesse est la même à un moment quelconque donné, dépend de 2 fonctions arbitraires de 3 arguments lorsque Φ_3 est donnée; si les forces massiques ne sont point prescrites, alors cette classe dépend de 1 fonction arbitraire de 4 arguments (on a attaché au système (Z) les équations $q_i = 0, (i = 1, 2, 3)$, qui caractérisent cette classe dans la classe générale (1)).

Pour les 3 classes particulières précédentes, le système (Z) fermé comprend 14 fonctions inconnues et les caractères réduits sont $s_0 = 6, s_1 = 3, s_2 = s_3 = 2$.

5. La classe particulière de la classe (1), dont les mouvements se développent tel que la direction de la vitesse reste la même à tous les points de D d'une manière permanente, dépend de 2 fonctions arbitraires de 3 arguments lorsque Φ_3 est donnée; si les forces massiques ne sont point prescrites, alors cette classe dépend de 1 fonction arbitraire de 4 arguments (on a attaché au système (Z) les équations $q_i = 0, q_i = 0, (i = 1, 2, 3)$, qui caractérisent cette classe dans la classe (1); le système (Z) fermé comprend 13 fonctions inconnues et les caractères réduits sont $s_0 = 6, s_1 = s_2 = s_3 = 2$).

6. La classe particulière de la classe (1) dont les mouvements se développent tel que $v = \text{const.}$ dépend de 2 fonctions arbitraires de 3 arguments lorsque Φ_1 est donnée; si les forces massiques ne sont pas données, alors cette classe dépend de 1 fonction arbitraire de 4 arguments.

7. La classe particulière de la classe (1) qui comprend les mouvements d'un fluide homogène ($\rho = \text{const.}$) dépend de 2 fonctions arbitraires de 3 arguments si Φ_1 est donnée; lorsque les forces massiques ne sont pas prescrites, alors cette classe dépend de 1 fonction arbitraire de 4 arguments.

Pour les 2 classes particulières précédentes, le système (Z) fermé comprend 12 fonctions inconnues et les caractères réduits sont $s_0 = 5$, $s_1 = s_2 = s_3 = 2$.

8. La classe (2, 3) (c'est-à-dire la classe particulière de la classe (1), dont les transformations se développent tel que la densité aussi que la grandeur de la vitesse ont des valeurs qui ne dépendent pas du point de D , à un moment quelconque donné) dépend de 1 fonction arbitraire de 3 arguments (par exemple q_3 , qui est la courbure de la ligne de courant).

9. La classe (2, 4) dépend de 1 fonction arbitraire de 3 arguments (par exemple ρ_2).

10. La classe (3, 4) dépend de 1 fonction arbitraire de 3 arguments (v_2).

Pour les précédentes 3 classes particulières de la classe (1), le système (Z) fermé comprend 11 fonctions inconnues et les caractères réduits sont $s_0 = 6$, $s_1 = 3$, $s_2 = s_3 = 1$.

11, 12. Chacune des classes (2, 5) et (3, 5) dépend de 1 fonction arbitraire de 3 arguments (par exemple ρ_2 , respectivement v_2); le système (Z) pour chacune de ces deux classes comprend 10 fonctions inconnues et les caractères réduits sont $s_0 = 6$, $s_1 = 2$, $s_2 = s_3 = 1$.

13...16. Chacune des classes (2, 7), (3, 6), (4, 6), (4, 7) dépend de 1 fonction arbitraire de 3 arguments (par exemple q_3 pour les premières 2 classes et respectivement ρ_2 et v_2 pour les 2 dernières classes); pour chacune de ces classes, le système (Z) fermé comprend 9 fonctions inconnues et les caractères réduits sont $s_0 = 5$, $s_1 = 2$, $s_2 = s_3 = 1$.

17, 18. Chacune des classes (5, 6), (5, 7) dépend de 1 fonction arbitraire de 3 arguments (ρ_2 , respectivement v_2); pour chacune de ces classes le système (Z) fermé comprend 8 fonctions inconnues et les caractères réduits sont $s_0 = 5$, $s_1 = s_2 = s_3 = 1$.

19. La classe (6, 7) dépend aussi de 1 fonction arbitraire de 3 arguments (q_3); le système (Z) fermé respectif comprend 7 fonctions inconnues et les caractères réduits sont $s_0 = 4$, $s_1 = s_2 = s_3 = 1$.

20. La classe (2, 3, 4), équivalente à la classe (2, 4, 7), dépend de 2 fonctions arbitraires de 1 argument (v_i , q_i); le système (Z) fermé comprend 6 fonctions inconnues et les caractères réduits sont $s_0 = 4$, $s_1 = 2$, $s_2 = s_3 = 0$.

21. La classe (2, 3, 5), équivalente à la classe (2, 5, 7), dépend de 1 fonction arbitraire de 1 argument (v_i); le système (Z) fermé comprend 5 fonctions inconnues et les caractères réduits sont $s_0 = 4$, $s_1 = 1$, $s_2 = s_3 = 0$.

22. La classe (3, 4, 6), équivalente à la classe (4, 6, 7), dépend de 1 fonction arbitraire de 1 argument (q_i); le système (Z) fermé comprend 2 fonctions inconnues et les caractères réduits sont $s_0 = 1$, $s_1 = 1$, $s_2 = s_3 = 0$.

23. La classe (3, 5, 6), équivalente à la classe (5, 6, 7), dépend de 3 constantes arbitraires.

Observations. 1. On a évité les prolongements du 2-ème ordre en construisant la chaîne des éléments intégraux

$$e_1 (\omega^2 = \omega^3 = dt = 0, \quad dv = v'_1 \omega^1, \quad d\rho = \rho'_1 \omega^1, \text{ etc}),$$

$$e_2 (\omega^3 = dt = 0, \quad dv = v'_1 \omega^1 + v'_2 \omega^2, \text{ etc}),$$

$$e_3 (dt = 0, \quad dv = v'_1 \omega^1 + v'_2 \omega^2 + v'_3 \omega^3, \text{ etc})$$

$$e_4 (dv = v'_1 \omega^1 + v'_2 \omega^2 + v'_3 \omega^3 + v'_4 dt, \text{ etc})$$

pour établir l'involution des systèmes (Z) correspondant aux classes (1), (5), (7); en construisant la chaîne

$$e_1 (\omega^1 = \omega^2 = \omega^3 = 0, \quad dv = v'_i dt, \quad d\rho = \rho'_i dt, \text{ etc}),$$

$$e_2 (\omega^2 = \omega^3 = 0, \quad dv = v'_i dt + v'_1 \omega^1, \text{ etc}),$$

$$e_3 (\omega^3 = 0, \quad dv = v'_i dt + v'_1 \omega^1 + v'_2 \omega^2, \text{ etc}),$$

$$e_4 (dv = v'_i dt + v'_1 \omega^1 + v'_2 \omega^2 + v'_3 \omega^3, \text{ etc})$$

pour établir l'involution des systèmes (Z) correspondants aux classes (2), (3), (4), (6), (2, 4), (3, 4), (2, 5), (3, 5), (4, 6), (4, 7), (5, 6), (5, 7); en construisant la chaîne

$$e_1 (\omega^2 = \omega^3 = \omega^1 = 0, \quad dv = v'_i dt, \text{ etc})$$

$$e_2 (\omega^3 = \omega^1 = 0, \quad dv = v'_i dt + v'_2 \omega^2, \text{ etc})$$

$$e_3 (\omega^1 = 0, \quad dv = v'_i dt + v'_2 \omega^2 + v'_3 \omega^3, \text{ etc}),$$

$$e_4 (dv = v'_i dt + v'_2 \omega^2 + v'_3 \omega^3 + v'_1 \omega^1, \text{ etc})$$

pour établir l'involution des systèmes (Z) correspondants aux classes (2, 3), (2, 7), (3, 6), (6, 7), et en construisant une chaîne quelconque du type:

$$e_1 (\omega^i = \omega^j = \omega^k = 0, \quad dv = v'_i dt, \dots),$$

$$e_2 (\omega^j = \omega^k = 0, \quad dv = v'_i dt + v'_i \omega^i, \dots),$$

$$e_3 (\omega^k = 0, \quad dv = v'_i dt + v'_i \omega^i + v'_j \omega^j, \dots),$$

$$e_4 (dv = v'_i dt + v'_i \omega^i + v'_j \omega^j + v'_k \omega^k, \dots)$$

pour les autres classes.

2. Il n'y a rien à préciser dans les résultats précédents pour les valeurs $\chi = 1,67$, $\chi = 1,4$ etc., adoptées dans les recherches techniques des

constructions gazodynamiques, lorsqu'il s'agit d'un gaz monoatomique, respectivement biatomique, etc.

3. Dans les résultats 8—23, la classe particulière de la classe (1) soumise simultanément aux conditions auxquelles sont soumises les classes i et j , a été notée par (i, j) ; la classe particulière de la classe (1) soumise simultanément aux conditions auxquelles sont soumises les classes i, j et k , a été notée par (i, j, k) .

Reçue le 28 octobre 1970

Institut Polytechnique, Jassy,
Chaire de Mécanique théorique

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MIȘCĂRI POLITROPICE DE-A LUNGUL UNEI FAMILII DE PLANE PARALELE (I)

(Rezumat)

Rezultatele prezentate, asupra arbitrarității a 23 clase de transformări politropice de-a lungul unei familii de plane paralele, au fost obținute prin utilizarea unor sisteme diferențiale exterioare ce nu necesită prelungiri de ordinul al 2-lea pentru stabilirea involuției lor.

ZUR ERLANGUNG EINIGER STÜCKE MIT ELEKTRISCHEN UND MAGNETISCHEN POLEN AUS EINER MISCHUNG VON FERRITPULVER UND CARNAUBAWACHS¹⁾

VON

C. PASNICU, EUGEN OSADEȚ und EMIL LUCA

1. Vorwort

In einem vorhergehenden Werk zeigte man, dass einige magnetische Ferriten, in bestimmten Bedingungen, ausgesetzt elektrischen Feldern, in derselben Zeit, verhalten sich wie Elektreten so wie Magneten, aufweisend zwei elektrische und zwei magnetische Polen [1].

Die elektrischen Eigenschaften dieser Stücke mit vier Polen, unter solchen Bedingungen erhalten, haben eine kleine Intensität und Zeitdauer.

Für die Verbesserung der Eigenschaften, haben wir vorgenommen aus magnetischen Stoffen und Dielektriken, mit hervortretend elektrischen Eigenschaften, homogenere Mischungspulver zu erzielen, die später gleichzeitig oder nacheinander folgend, der Aktion elektrischen und magnetischen Feldern ausgesetzt zu werden.

In der vorliegenden Arbeit, haben wir Mischungen aus magnetischer Ferritpulver, Typ Ba + Bi₂O₃ [2], [3] und Carnaubawachspulver, wessen elektretische Eigenschaften bekannt sind, erhalten [4].

2. Experimentalisches Verfahren

Die Proben gebildet aus 25 % des feinen magnetischen Ferritenpulver, Types Ba + Bi₂O₃ und 75 % feinem Carnaubawachspulver, wurden durch die Realisation einer homogenen Mischung, in festem Zustand der zwei Konstituenten erhalten; nachdem sie in eine Matrize unter einem Druck von 500 at., eingeführt wurden.

¹⁾ Mittgeteilt an der wissenschaftlichen Tagung der Technischen Hochschule Jassy, am 3.—4. Juli 1970.

Man erhielt parallelipipedförmige Muster mit folgenden Dimensionen: $14 \times 14 \times 4$ mm.

Die ausgeführte Messungen, gleich nach der Formung der Proben, mit der Methode der elektrostatischer Induktion, mit Hilfe eines Elektrometers ORION von grosser Sensibilität, in Bezug auf die elektrische Ladung und in Bezug auf den magnetischen Zustand, mit einem von uns hergestellten magnetometrischer Vorrichtung, zeigten, dass diese, keine oberflächige Elektrizitätsladungen entstammt aus remanentlicher Polarisationsentwicklung und auch keine magnetische Remanenz, darbieten.

Für die Erlangung dieser Stücke mit einem Paar elektrischen und magnetischen Pol, die so erhaltete Proben wurden der Aktion eines Gleichstromfeldes von 5000 V/cm., und ein magnetischen Feldes von 0,9 Tesla, ausgesetzt.

Die Muster wurden für dies, in einem parallelipedischen Gefäss aus Plexiglas von annehmbaren Dimensionen, zwischen zwei Kupferelektroden die für das Anbringen des elektrisch polarisierendes Feldes dienten durch ihrer Verbindung mit der Hochspannungsquelle, so wie auch für die Versicherung des thermischen Kontaktes zwischen der Probe und die Wände des Gefässes eingeführt.

Dieses Plexiglasgefäss wurde in einem anderen Innenraum voll mit Öll eingesetzt, dessen Temperatur mit einem elektrischen Widerstand und einem Relais, akioniert mit einem Kontakttermometer regliert wurde.

Der so erhaltete Innenraum, wurde zwischen den Polen des Elektromagnetes der das Feld von 0,9 Tesla generierte, eingesetzt.

Während der langsamen Erwärmung der Proben, und der Bewahrung dieser, eine zeitweile von einer Stunde, unter der konstanter Temperatur von 75°C und nachher während ihrer langsamen Abkühlung bis zur Zimmertemperatur, in folgenden Konditionen, wurde mit einem elektrischen und einem magnetischen Feld akioniert:

1. Die Proben wurden nur der Aktion des elektrischen Feldes ausgesetzt.

2. Die Proben wurden gleichzeitig der Aktion des magnetischen und elektrischen Feldes, parallel orientiert, ausgesetzt.

3. Proben die gleichzeitig der Aktion eines magnetischen und elektrischen Feldes, antiparallel orientiert, ausgesetzt wurden.

4. Proben die gleichzeitig der Aktion eines magnetischen und elektrischen Feldes, senkrecht orientiert, ausgesetzt wurden.

Die so erhaltene Proben, bekommen Elektreteigenschaften so wie auch magnetische Eigenschaften.

Den elektrischen Zustand der remanentischer Polarisation, bestimmte man, durch die Messung der elektrischer Ladung auf die Seiten und den Zustand der magnetischen Remanenz durch die Bestimmung der Magnetisationsintensität mit einer magnetometrischer Vorrichtung.

Diese Messungen wurden in Zeit wiederholt, damit man die Stabilität der Proben bestimmen soll.

3. Experimentalische Ergebnisse

1. Für die Proben die nur der Aktion des magnetischen Feldes ausgesetzt wurden, stellt man nur eine magnetische Entwicklung vor, bestimmt von der Orientation des magnetischen Pulvers aus Ferrit, so wie auch ein Polarisationsprozess des Carnaubawachspulvers, mit Erscheinung, auf den Seiten die vor den Polen des Elektromagnetes gesetzt wurden, einigen elektrischen Ladungen [5]... [7] und [8].

Weil der magnetische Zustand sich konstant in Zeit behält, in der Abb. 1 zeigt man die variationskurve in Zeit der elektrischen Ladung von den beiden Seiten des Elektreten.

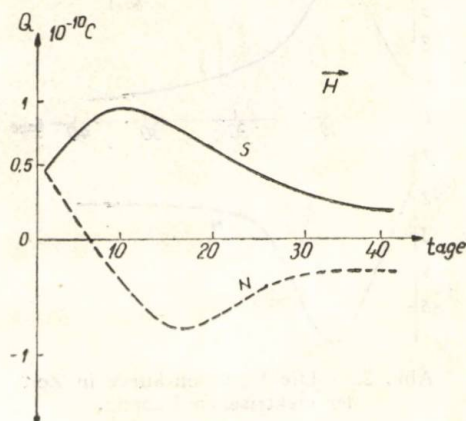


Abb. 1. — Die Variationskurve in Zeit der elektrischen Ladung.

Am Anfang, sind die elektrische Ladungen von den beiden Seiten gleich, and haben dasselbe Zeichen.

Nach einer Zeitdauer, die elektrische Ladung von der Seite der Probe, die dem Südpol des Elektromagneten entspricht, bewahrt sein Zeichen und verkleinert seinen Wärt: während die elektrische Ladung von der Seite die dem Nordpol des Elektromagneten entspricht, sein Zeichen wechselt und naccher verändert es sich ungefähr gleichzeitig, wie die von der entgegengesetzten Seite.

Das so erhaltene Stück, vorzeigt auf denselben Seiten magnetische Polen so vie auck elektrische Polen und benimmt sich als Magnet so wie als Elektret.

2). 3). Die Proben, die der gleichzeitiger Aktion eines elektrischen und magnetischen Feld untergeworfen wurden, die parallel oder ähnlicher antiparallel akionieren, kommt ein Magnetisationsprozess zum Vorschein, bestimmt von der Orientation des magnetischen Ferritpulvers so vie auch ein elektrischer Polarisationsprozess, zukommend von dem Carnaubawachs.

Diese mal stellt man fest, dass der Magnetisationszustand weniger stark wie im Vorhergehendem Fall ist, bewahrend sich konstant in Zeit, und der Polarisationszustand, durch den Wert von elektrischen Ladungen von den Seiten der Stüke, ausgedrückt, verändert sich während des Zeitens entsprechend der Abb. 2, wenn die zwei Felder parallel sind und Abb. 3 wenn sie antiparallel sind.

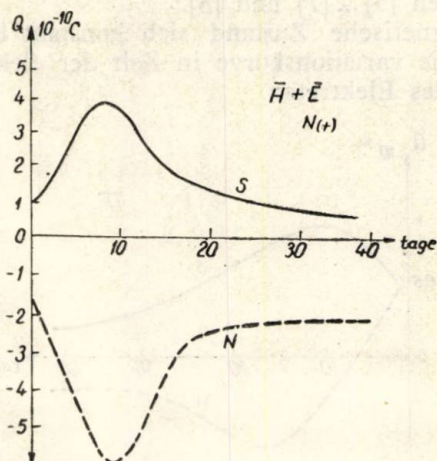


Abb. 2. — Die Variationskurve in Zeit der elektrischen Ladung.

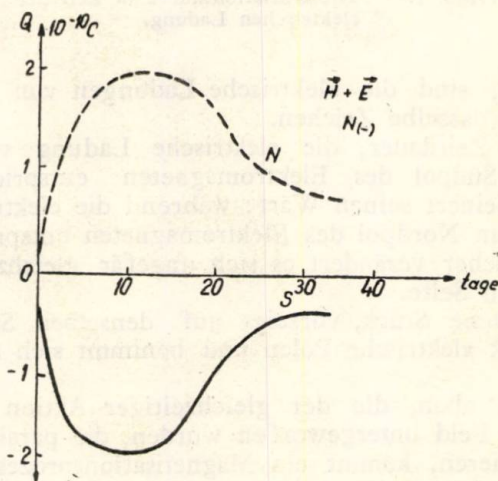


Abb. 3. — Die Variationskurve in Zeit der elektrischen Ladung.

Aus der Analyse dieser Kurven, folgt:

In Falle beider Proben, die elektrische Ladungen, die zum Vorschein auf den Seiten die den Polen des Elektromagnetes entsprechen kommen, sind gleich und haben entgegengesetzte Zeichen. Sie folgen denselben Variationsprozess in Zeit.

Die Zeichen der elektrischer Ladungen, die auf den Seiten des sogeformtes Elektret erscheinen, sind von der Aktionsmodalität der beiden Felder abhängig, parallel oder antiparallel; die positive Ladung erscheint auf der eintretender Seite der Kraftlinien des Elektrischen Feldes, also welche die dem positiven Zeichen dessen entspricht.

Aus den obigen folgt, dass auch in diesem Fall, das erlangte Stück sich wie Magnet so wie auch als Elektret benimmt; ein Stück mit vier Polen, — zwei magnetische und zwei elektrische Polen, darstellend.

4) Für die Proben, die der gleichrichtiger Aktion eines magnetischen und elektrischen Felden ausgesetzt wurden, aber die senkrecht orientiert wurden, sind die experimentalische Ergebnisse, auf die Variation in Zeit der elektrischen Ladungen hinblickend im Abb. 4 dargestellt.

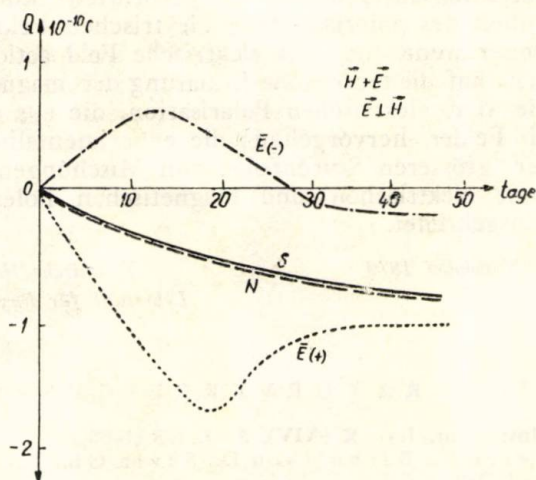


Abb. 4. — Die Variationskurve in Zeit der elektrischen Ladung.

Dieses mal stellt man fest, das neben dem gleichartigem Magnetisationsprozess de vorrigen Proben, auf den Seiten der Proben, elektrische Ladungen infolge der Aktion des magnetischen so wie auch des elektrischen Feldes vorhanden sind.

Die elektrische Ladungen, die in diesem Falle, auf den Seiten, entsprechend den Polen des Elektromagnetes eintreten, denselben Zeichen haben: ihr Wert wächst während des Zeitens.

Die elektrische Ladungen, die der Aktion des elektrischen Feldes zurückzuführen sind und welche auf den Seiten der Probe, auf welche dieses Feld actioniert, erscheinen, haben Anfänge entgegengesetzte Zeichen: nachher bekommen sie dasselbe Zeichen.

4. Schlussfolgerungen

Wenn man, unter bestimmten Bedingungen, magnetische Ferritenpulver und Carnaubawachs vermischt, kann man Stücke erlangen, die unter der Aktion eines magnetischen Feldes oder eines elektrischen Feldes verbunden mit einem elektrischen Feld, gleichzeitig Magneten so wie auch Elektreten werden: sie haben also zwei magnetische und zwei elektrische Polen.

Die Stellung der elektrischen Polen, betreffend die Aufladung der beiden Seiten die das Elektret charakterisieren, mit positiven oder negativen elektrischen Ladungen, hängt von der Orientation des magnetischen Feldes, unter deren Aktion sich die beiden Polenpaaren formieren.

Das Magnetisationszustand der so geformten Stücke, hängt auch von der Anwesenheit des polarisatischen elektrischen Feld; die Magnetisation ist viel geringer wenn auch das elektrische Feld actioniert.

In Anbetracht auf die teoretische Erklärung der magnetischen Prozesse so wie auch die der elektrischen Polarisation, die aus der gleichzeitiger Aktion der zwei Felder hervorgehen; die experimentalische Forschungen werden bei einer grösseren Stufenfolge von Mischungen, aus denen man solche Stücke mit elektrischen und magnetischen Polen erlangen kann, fortgesetzt und ausgebreitet.

Eingegangen am 10. November 1970

Technische Hochschule, Jassy,
Lehrstuhl für Physik und Elektronik

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ASUPRA OBTINERII UNOR PIESE CU POLI ELECTRICI ȘI MAGNETICI DIN AMESTEC DE PULBERI DE FERITĂ ȘI CEARĂ DE CARNAUBA

(Rezumat)

În scopul obținerii unor piese care să aibă în același timp atât proprietăți de electret cât și de magnet, s-au realizat amestecuri din 25% pulberi fine de ferită magnetică de $Ba + Bi_2O_3$ și 75% pulberi de ceară de Carnauba, supuse apoi unei presiuni de cca. 500 at.

Probele astfel obținute și încălzite la temperaturi de $80^\circ C$ au fost supuse: acțiunii unui câmp magnetic; acțiunii simultane a unor câmpuri electric și magnetic orientate paralel; acțiunii simultane a unor câmpuri electric și magnetic orientate antiparalel; acțiunii simultane a unor câmpuri electric și magnetic orientate perpendicular.

În toate cazurile, în probele utilizate au apărut atât procese de polarizare remanentă, în urma înghețării dipolilor orientați, cât și procese de magnetizare remanentă.

S-au obținut astfel piese ce se comportă atât ca electret cât și ca magnet, având deci atât poli electrici cât și poli magnetici, a căror poziție depinde de modul de orientare a câmpurilor electric și magnetic care au produs procesele de polarizare și magnetizare.

THE UNIVERSITY OF CHICAGO
DEPARTMENT OF CHEMISTRY
CHICAGO, ILLINOIS

TO THE HONORABLE CHIEF OF BUREAU OF MINES
WASHINGTON, D. C.

SIR:

I have the honor to acknowledge the receipt of your letter of the 10th inst. and in reply to inform you that the same has been forwarded to the proper authorities for their consideration.

I am, Sir, very respectfully,
Yours very truly,
J. H. MANNING

SUR LES PROPRIÉTÉS DIÉLECTRIQUES DE QUELQUES PRODUITS POLYESTERIQUES ¹⁾

PAR

VLAD CEHAN, EUGEN NEAGU, GHEORGHE ANTONIU et EMIL LUCA

1. Considérations générales

La production des polyesters qui ont des propriétés mécaniques et chimiques remarquables, a soulevé le problème de leurs propriétés diélectriques dans différentes conditions de travail.

Tout comme dans des travaux antérieurs, concernant d'autres produits ([1] ... [4]), nous nous sommes proposés d'étudier, dans le cas des produits polyesteriques, la variation de la permittivité ϵ_r et des pertes diélectriques exprimées par $\text{tg } \delta$, en fonction de la fréquence et de la température.

On a étudié les produits polyesteriques suivants: bande de téréphtalate de polyéthylène, filaments bruts récoltés après tirage de la filière, sans apprêt antistatique, et filaments de laine synthétique avec apprêt antistatique.

2. Résultats expérimentaux

Nous avons utilisé une cellule de mesure réalisée dans le but de la recherche. Avant de les y introduire, les filaments ont été coupés en petits morceaux.

Les mesures expérimentales de ϵ_r et de $\text{tg } \delta$ ont été effectuées à l'aide d'un pont TESLA TM 351. On a fait varier la température de 20°C jusqu'à 130°C en introduisant la cellule dans un bain à l'huile.

2.1. Variation de ϵ_r et de $\text{tg } \delta$ en fonction de la fréquence

En vue des mesures expérimentales on a utilisé une gamme de fréquences comprise entre 50 et 10 000 Hz. Les mesures ont été effectuées à la tempé-

1) Présentée lors de la session scientifique de l'Institut polytechnique de Jassy, le 3—5 juillet 1970.

rature de 20°C, sur les preuves préalablement chauffées jusqu'à 130°C et ensuite lentement refroidies.

Les résultats expérimentaux sont donnés dans les fig. 1...3.

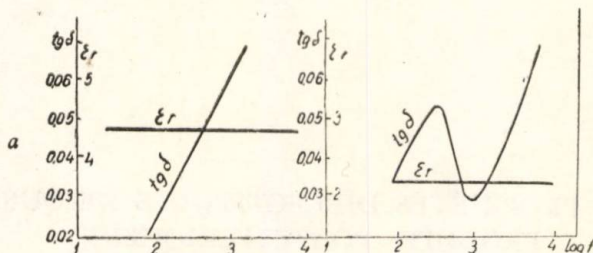


Fig. 1. — Variation de ϵ_r et de $\operatorname{tg} \delta$ en fonction de la fréquence pour la bande de téréphtalate de polyéthylène. *a*—preuves non chauffées; *b*—preuves chauffées jusqu'à 130°C et refroidies.

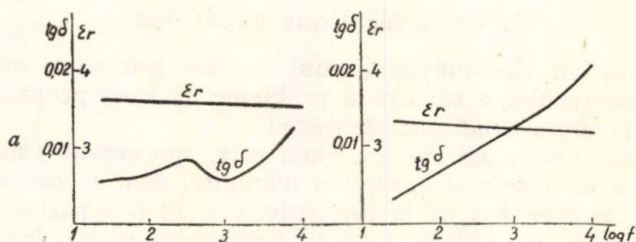


Fig. 2. — Variation de ϵ_r et de $\operatorname{tg} \delta$ en fonction de la fréquence pour les filaments bruts, sans apprêt antistatique. *a*—preuves non chauffées; *b*—preuves chauffées jusqu'à 130°C et refroidies.

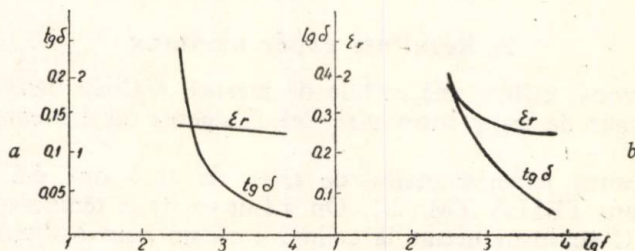


Fig. 3. — Variation de ϵ_r et de $\operatorname{tg} \delta$ en fonction de la fréquence pour les filaments de laine, avec apprêt antistatique. *a*—preuves non chauffées; *b*—preuves chauffées jusqu'à 130°C et refroidies.

On constate une légère diminution, presque linéaire, de la permittivité, celle-ci ayant une valeur plus grande dans le cas des preuves soumises au traitement thermique.

Les pertes diélectriques exprimées par $\text{tg } \delta$ varient différemment pour les preuves soumises au traitement thermique par rapport au preuves non traitées, dépendant aussi du fait si elles ont été traitées par l'apprêt antistatique ou non. Les pertes de la bande de téréphtalate de polyéthylène et des filaments sans apprêt antistatique présentent un maximum pour la fréquence de 400 Hz et un minimum pour 1000 Hz, qu'on ne constate pas aux filaments de laine avec apprêt.

Nous sommes d'avis que l'apparition de ces maxima et minima est due aux transitions de la phase qui ont lieu pendant le traitement thermique. Dans le cas de la preuve de laine avec apprêt antistatique il n'a pas été possible de mettre en évidence les transitions de la phase à cause de la conductivité élevée qui dissimule le processus.

2.2. Variation de ϵ_r et de $\text{tg } \delta$ en fonction de la température

En vue de mettre en évidence les transitions de la phase qui ont lieu pendant le traitement thermique on a tracé les courbes de la variation de ϵ_r et de $\text{tg } \delta$ pour les fréquences de 400 Hz et de 10000 Hz pour une variation de la température de 20°C jusqu'à 130°C.

Les résultats expérimentaux sont donnés dans les fig. 4...6.

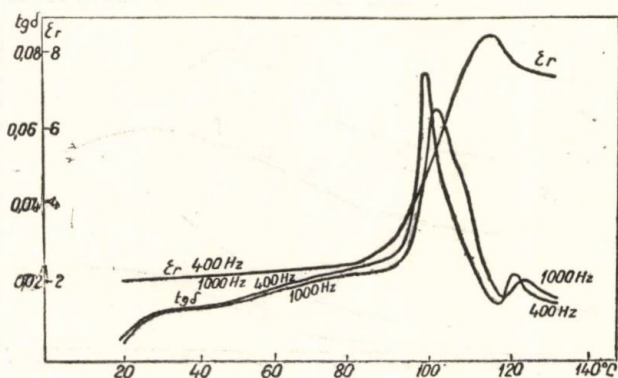


Fig. 4. — Variation de ϵ_r et de $\text{tg } \delta$ en fonction de la température pour la bande de téréphtalate de polyéthylène.

L'analyse des courbes tracées met en évidence des transitions de la phase pour les preuves de téréphtalate de polyéthylène et pour les filaments sans apprêt, qui se manifeste par une augmentation accentuée de ϵ_r et de $\text{tg } \delta$.

À cause de l'agent antistatique qui a une grande conductivité, les preuves de laine avec apprêt présentent des maxima et des minima moins importants, les transitions de la phase étant moins évidentes.

En général on constate que les maxima de ϵ_r et de $\operatorname{tg} \delta$ ne correspondent pas à la même valeur de la température.

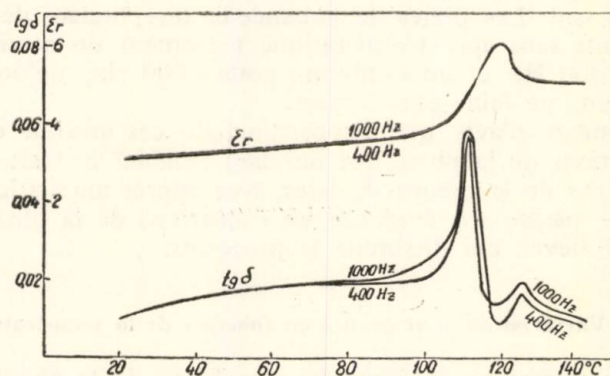


Fig. 5. — Variation de ϵ_r et de $\operatorname{tg} \delta$ en fonction de la température pour les filaments bruts, sans apprêt anti-statique.

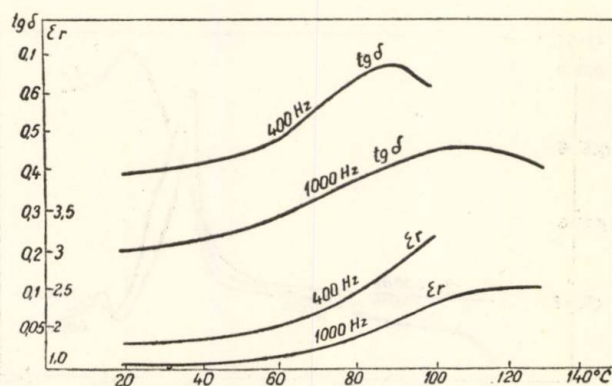


Fig. 6. — Variation de ϵ_r et de $\operatorname{tg} \delta$ en fonction de la température pour les filaments de laine avec apprêt anti-statique.

Ce fait, de même que l'existence d'un second maximum pour $\operatorname{tg} \delta$ à une température plus élevée, imposent, en vue d'une interprétation théorique, l'extension des mesures expérimentales sur d'autres produits polyesteriques.

Reçue le 10 novembre 1970

Institut Polytechnique, Jassy,
Chaire de Physique et d'Électronique

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ASUPRA PROPRIETĂȚILOR DIELECTRICE ALE UNOR
PRODUSE POLIESTERICE

(Rezumat)

Se determină valorile permitivității ϵ_r și ale pierderilor dielectrice, exprimate prin $\operatorname{tg} \delta$, la unele produse poliesterice românești, anume bandă de polietilen tereftalat, fibre brute trase direct din filieră, netratate antistatic, și fibre de lână sintetică, tratate antistatic. S-a urmărit variația acestor parametri cu frecvența și cu temperatura în condițiile în care epruvetele au fost supuse sau nu, în prealabil, unui tratament de încălzire pînă la 120°C.

Rezultatele experimentale denotă că în urma procesului de încălzire apar schimbări de fază, puse în evidență prin variații accentuate (maxime) ale lui ϵ_r și $\operatorname{tg} \delta$, care permit determinarea temperaturii la care aceste schimbări de fază au loc.

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BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI

TOMUL XVII (XXI)

Fasc. 3—4

**SECTIA I MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

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UN CALCUL DES APPLICATIONS

PAR

AL. CLIMESCU

1. Soit F l'ensemble des applications (univoques) $f: M \rightarrow N$; il est facile de doter F d'une structure algébrique lorsque R possède une telle structure. Par exemple (R étant l'ensemble des réels, $R^2 = R \times R$ etc.) on a :

Si $N = R$,

$$(f \vee g)(x) = \max [f(x), g(x)],$$

$$(f \wedge g)(x) = \min [f(x), g(x)]$$

et F devient un treillis distributif.

Par ordination lexicographique R^2 devient un treillis distributif si $N = R^2$, F est aussi un treillis distributif.

Si $N = R$, on peut poser

$$(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n) \Leftrightarrow (\forall i) a_i \leq b_i$$

et encore (k réel quelconque)

$$(a_1, a_2, \dots, a_n) + k = (a_1 + k, a_2 + k, \dots, a_n + k)$$

N devient un d -treillis et f de même (pour l'intéressante théorie des d -treillis v. [2]).

Lorsque N est amorphe il semble à première vue qu'on doive laisser F amorphe aussi, mais nous voulons justement montrer qu'il est possible de le munir d'une opération unaire et de plusieurs opérations binaires et cela avec un large degré d'arbitraire, la structure algébrique ainsi attribuée à F étant analogue au calcul des propositions.

2. Proposition I. Si P n'est pas vide, on peut toujours y introduire une opération binaire qui le transforme en un semi-treillis.

Si P est fini c'est bien connu. Si P est infini, cela peut se faire au moins de deux manières différentes.

a). Soit S un semi-treillis connu, Π une partition de P avec $\text{card } \Pi = \text{card } S$, $A, B, \dots$ les classes de Π ; choisissons dans chaque classe un élément distingué $a, b, \dots$ et posons :

$xx = x$; $(\forall x \forall y)(x \neq y), x \in A, y \in A, xy = a$; $(\forall x)(\forall y)x \in A, y \in B \quad xy =$ l'élément distingué de la classe AB ; la commutativité et l'idempotence sont évidentes et la vérification de l'associativité est un calcul de routine.

b) Soient $M_0 = \{0\}$, $M_1 M_2 \dots M_k \neq \emptyset$ et P l'ensemble des mots formé à partir de la suite des M par les deux règles suivantes :

1° 0 est le seul mot de longueur nulle.

2° Si m est un mot de longueur i , $m x$ ($x \in M_{i+1}$) est un mot de longueur $i + 1$.

$\text{Card } P$ peut être évidemment quelconque et il devient un semitreillis en définissant l'intersection de deux mots comme le mot formé des symboles communs aux deux mots, lus de gauche à droite.

Cette méthode inspirée par les graphes arborescents, peut être diversifiée de bien de manières (suites dénombrable ou même transfinies d'ensembles M , restrictions à la formation des mots de longueur immédiatement supérieure etc).

Proposition II. *L'ensemble F des applications $f: M \rightarrow N$ (univoques et partout définies peut toujours être algébriquement structuré comme un calcul analogue au calcul des propositions (N infini ou fini a $2n$ éléments).*

On arrive au résultat par une suite de constructions et de définitions.

a) Partageons N en deux parties P, Q disjointes et équipotentes $\alpha: P \rightarrow Q$ réalisant la bijection.

b) Définissons dans N l'opération unaire notée $\bar{a}$, par

$$\bar{a} = \alpha(a) \text{ si } a \in P, \bar{a} = \alpha^{-1}(a) \text{ si } a \in Q.$$

C'est une opération involutive partout définie dans N .

c) Transformons P en un semi-treillis (opération notée ab) et posons ($a, b \in P$) $\alpha(ab) \stackrel{\text{def}}{=} \alpha(a) \alpha(b)$ (ou $\overline{ab} = \bar{a} \bar{b}$). Q devient un semi-treillis isomorphe à P .

d) Définition de la négation $(\bar{f})(x) = \overline{f(x)}$.

e) Définition de la disjonction

$$(f \vee g)(x) = \begin{cases} f(x) g(x) & \text{lorsque } f(x), g(x) \in P \\ f(x) & \text{lorsque } f(x) \in P, g(x) \in Q \\ g(x) & \text{lorsque } f(x) \in Q, g(x) \in P \\ f(x) g(x) & \text{lorsque } f(x), g(x) \in Q. \end{cases}$$

f) Définition des autres opérations (conjonction, implication, équivalence)

$$f \wedge g = \overline{\bar{f} \vee \bar{g}}, f \rightarrow g = \bar{f} \vee g \quad (f \equiv g) = (f \rightarrow g) \wedge (g \rightarrow f) = (\bar{f} \vee g) \wedge (\bar{g} \vee f).$$

g) Appelons vrais les éléments de P et faux les éléments de Q ; appelons aussi f tautologie lorsque toutes ses valeurs sont vraies; on a les égalités:

$$\bar{f} = f, f \vee f = f, f \vee g = g \vee f, f \vee (g \vee h) = (f \vee g) \vee h, f \wedge f = f, \\ f \wedge g = g \wedge f, f \wedge (g \wedge h) = (f \wedge g) \wedge h$$

et les tautologies

$$f \equiv f \vee (f \wedge g), f \equiv f \wedge (g \vee f), f \wedge (g \vee h) \equiv (f \wedge g) \vee (f \wedge h), \\ f \vee (g \wedge h) \equiv (f \vee g) \wedge (f \vee h), f \rightarrow f \vee g, (f \rightarrow g) \rightarrow (h \vee f \rightarrow h \vee g).$$

À titre d'exemple choisissons la tautologie

$$(f \rightarrow g) \rightarrow (h \vee f \rightarrow h \vee g)$$

notée S . Pour que S ne soit pas une tautologie, il faudrait qu'il y ait un x pour lequel

$$(f \rightarrow g)(x) \in P \text{ et } [h \vee f \rightarrow h \vee g](x) \in Q,$$

c'est-à-dire $(h \vee f)(x) \in P, (h \vee g)(x) \in Q, (\bar{f} \vee g)(x) \in P$, ou encore $h(x) \in Q, g(x) \in Q, f(x) \in P$ et $(\bar{f} \vee g)(x) \in Q$, ce qui est contradictoire.

En observant qu'une égalité $X = Y$ est a fortiori une tautologie $X \equiv Y$, on reconnaît parmi les tautologies indiquées les axiomes de Hilbert-Ackermann [1, p. 23] pour le calcul des propositions; il s'en suit que tous les moyens disponibles dans ce calcul se transmettent ici—vérification des tautologies par la méthode des matrices, formes normales, application de la méthode des tableaux de Beth-Hintikka-Smullyan [5] et que les théorèmes du calcul des propositions restent valables; en particulier toute tautologie du calcul des propositions reste une tautologie dans le sens indiqué ici.

On a encore la

Proposition III. *La relation r entre applications définie par $f r g \Leftrightarrow f \equiv g$ est une tautologie est une équivalence et les classes de cette équivalence forment par les opérations de disjonction, conjonction et négation entre classes une algèbre de Boole isomorphe avec l'algèbre de Boole de toutes les parties de N .*

La dernière assertion résulte du fait que chaque classes peut être représentée par une application $N \rightarrow \{0,1\}$.

3. Chaque tautologie $X \equiv Y$ donne lieu à l'équation fonctionnelle $X = Y$. On a la

Proposition IV. *L'équation $X = Y$ ou $X \equiv Y$ est une tautologie peut être résolue par voie algorithmique.*

Par la méthode des matrices on calcule les valeurs de X et Y en chaque point (ces valeurs sont par hypothèse vraies ou fausses en même

temps) et l'égalité de ces valeurs conduit à des égalités dans les treillis P et Q qui montrent quelles sont les conditions à imposer aux fonctions figurant dans la tautologie.

Exemple: $f = f \wedge (f \vee g)(a, b \dots \in P, \bar{a}, \bar{b}, \dots \in Q)$

$f(x)$	$g(x)$	$(f \vee g)(x)$	$[f \wedge (f \vee g)](x)$
a	b	ab	ab
a	$\bar{b}$	a	a
$\bar{a}$	b	b	$\bar{a}$
$\bar{a}$	$\bar{b}$	$\bar{a} \bar{b}$	$\bar{a} \bar{b}$

On en déduit que l'équation est satisfaite pour tout couple qui satisfait pour tout x à l'une des conditions suivantes:

- 1) $f(x) \in P$ et $g(x) \in P$ avec $f(x)g(x) = f(x)$ ou
- 2) $f(x) \in P$ et $g(x) \in Q$ ou
- 3) $f(x) \in Q$ et $g(x) \in P$ ou
- 4) $f(x) \in Q$ et $g(x) \in Q$ avec $f(x)g(x) = f(x)$.

Il est clair que lorsque $N = \{0, 1\}$ toutes les tautologie $X \equiv Y$ deviennent des égalités et nous émettons la conjecture que si $\text{card } N > 2$, ce fait est impossible.

4. La construction précédente s'applique à toutes les applications partout définies. Pour un calcul des applications qui ne satisfont pas à cette condition, il paraît être nécessaire d'accepter les applications multivoques à un nombre fini de valeurs de N et plus précisément de poser $f(x) = \{F_x, F'_x\}$, F_x, F'_x parties finies disjointes de N . Ces applications seront appelées multivoques à caractère fini et leur ensemble sera désigné par F . On peut identifier $f(x) \in a$ à $f(x) = \{\{a\}, \emptyset\}$ et $f(x) = \{\emptyset, \emptyset\}$ sera interprété comme le fait que $f(x)$ n'est pas définie en x .

On a la

Proposition V. F devient un calcul analogue au calcul des propositions avec les définitions suivantes:

$$\bar{f}(x) = (F'_x, F_x), (f \vee g)(x) = (F_x \vee G_x, F'_x \wedge G'_x)$$

$$(f \wedge g)(x) = \overline{f \vee g}(x), (f \rightarrow g)(x) = (f \vee g)(x)$$

$$(f \equiv g)(x) = [(f \rightarrow g) \wedge (g \rightarrow f)](x).$$

Les tautologies seront les f avec $F'_x = \emptyset$, les contradictions seront caractérisées par $F_x = \emptyset$.

Avec ces définitions les lois de distributivité et d'absorption deviennent des égalités, donc la

Proposition VI. *Par rapport à la conjonction et disjonction F est un treillis qui admet comme sous-treillis (ismorphes) celui des tautologies et celui des contradictions.*

5. Remarques. 1) α détermine dans N une involution sans points fixes; si l'on accepte de tels points, on peut construire un calcul des applications analogue à une logique à plusieurs et même à une infinité de valeurs de vérité. En particulier s'il n'y a qu'un seul point fixe, le calcul obtenu sera analogue à la logique de Lukasiewicz à trois valeurs.

2) Le calcul logique et la notion d'application sont liés encore d'une manière: on peut décrire le comportement de f en x en construisant d'abord l'ensemble des propositions

f n'est pas définie en x

$f(x) = a$

$f(b) = b$

.....

et en affirmant qu'une proposition et une seule est vraie de cet ensemble, ce qui nous conduit à la définition suivante:

Une fonction est une famille indexée $M_i, i \in I$ d'ensembles de propositions, chaque M contenant une seule proposition vraie. Pour que cette définition devienne utile il faut naturellement des conditions supplémentaires, mais leur recherche dépasse le cadre de cette Note.

3) Depuis quelques années l'algèbre des fonctions est redevenue d'actualité; parmi les travaux consacrés à cette question citons les très intéressantes recherches de B. Schweizer et A. Sklar [3] et [4].

Reçue le 25 mars 1971

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UN CALCUL AL APLICĂȚIILOR

(Rezumat)

Cu ajutorul unor construcții și definiții potrivite se arată posibilitatea construcției unui calcul al aplicațiilor peste tot definite, analog calculului propozițiilor (propoziția II). De asemenea (propoziția V) se construiește un calcul al aplicațiilor multivoce de caracter finit (§ 4) în cadrul căruia aplicațiile univoce nu peste tot definite se încadrează în mod natural.

PLANS AFFINES ET PLANS DE MÖBIUS

PAR

BERNARD D'ORGEVAL

1. Une structure d'incidence finie est caractérisée par la donnée de deux ensembles $\mathcal{P}$ contenant p éléments, et $\mathcal{B}$ contenant b éléments (p et b entiers positifs) et d'une fonction $\mathcal{I}$ définie sur $\mathcal{P} \times \mathcal{B}$ prenant les valeurs 0 ou 1. Si l'on appelle P_i et B_k deux éléments appartenant l'un à $\mathcal{P}$, l'autre à $\mathcal{B}$, si $\mathcal{I}(P_i, B_k) = 1$, on dit que P_i est incident à B_k , ou que B_k est incident à P_i . Généralement un élément P est dit point, un élément de $\mathcal{B}$ est dit bloc. Dans certaines structures particulières les blocs sont appelés „droites“, „cercles“.... On dit ainsi que P_i appartient au bloc B_k ou que le bloc B_k passe par le point P_i .

À toute structure d'incidence, on peut associer une matrice dite d'incidence (a_{ik}) de type $p \times b$, avec $a_{ik} = \mathcal{I}(P_i, B_k)$. À cette matrice se lie le nombre $N(\mathcal{A})$ des éléments de la matrice égaux à l'unité. Reiman [1] a démontré, que si deux points distincts sont incidents à au plus t blocs:

$$(1) \quad N(\mathcal{A}) \leq \frac{1}{2} (b + \sqrt{b^2 + 4bp(p-1)t}),$$

l'égalité ne valant que si chaque bloc contient le même nombre de points $s = N(\mathcal{A})/b$; dans ce cas, par deux points passent exactement t blocs.

2. M. U. Oliveri [2] a démontré qu'une structure $\mathcal{A} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ telle que $b = q^2 + q$ (q entier supérieur à 1), $p = q^2$, dont deux blocs ont au plus un point commun, et où $N(\mathcal{A}) = q^2(q+1)$ est un plan affine fini irréductible d'ordre q .

La Signorina P. Biscarini a démontré que une structure d'incidence $\mathcal{M} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ telle que $\alpha)$ $p = q^2 + 1$, $\beta)$ $b = q(q^2 + 1)$, $\gamma)$ deux points distincts appartiennent à au plus $q + 1$ blocs, $\delta)$ $N(\mathcal{M}) = q(q+1)(q^2+1)$ est un plan de Möbius fini d'ordre q , les blocs en sont les cercles contenant exactement $q + 1$ points.

3. Nous allons d'abord vérifier que l'imposition d'un point au sens de B. Segre [4] à un plan de Möbius détermine bien un plan affine. Nous avons donc une structure $\mathcal{M}$ représentée par une matrice à $q^2 + 1$ lignes et $q(q^2 + 1)$ colonnes formée de 1 et 0 et vérifiant $\sum_{i,k} m_{ik} = q(q+1) \times (q^2 + 1)$; $\sum_k m_{ik} m_{jk} = q + 1$, pour tout couple i, j distincts. Imposons le point P_1 ; nous ne conserverons dans la nouvelle structure que les blocs contenant ce point, puis nous retirerons ce point. La nouvelle structure aura une matrice à q^2 lignes et b' colonnes? Cherchons à calculer b' ; soit P_i un point différent de P_1 ; ensemble ils définissent $q + 1$ blocs, or on peut choisir q^2 points P_i et chacun de ces blocs peut être défini q fois par ce procédé, donc $b' = q^2(q + 1)/q = q^2 + q$. Les colonnes enlevées de la matrice sont donc $b - b' = q^3 - q^2$; chacune de ces colonnes contient $q + 1$ unités. La matrice de la structure obtenue contient donc N' unités avec

$$(2) \quad N' = q(q + 1)(q^2 + 1) - q(q + 1) - (q^3 - q^2)(q + 1) = q^2(q + 1).$$

Comme trois points du plan de Möbius définissent un seul cercle, deux points de la nouvelle structure n'y définiront qu'une droite et nous sommes dans le cas d'Oliveri. La structure finale est bien un plan affine.

Nous retrouvons bien le résultat connu que l'imposition d'un point à un plan de Möbius en fait un plan affine.

4. Nous allons maintenant montrer que réciproquement la connaissance d'une structure de plan affine permet dans certaines conditions d'obtenir un plan de Möbius.

Nous supposons donc connu un plan affine, c'est-à-dire une matrice à q^2 lignes et $q^2 + q$ colonnes, telle que le nombre $N(\mathcal{A}) = q^2(q + 1)$ et que pour tout i, j distinct $\sum_k a_{ik} a_{jk} = 1$. Si nous voulons par le procédé inverse du précédent obtenir un plan de Möbius, nous devons construire une matrice $\mathcal{M}$ à $q^2 + 1$ lignes et $q^3 + q$ colonnes avec $N(\mathcal{M}) = q(q + 1)(q^2 + 1)$ et pour tout i, j distincts $\sum_k a_{ik} a_{jk} = q + 1$.

Nous commencerons par ajouter une ligne à la matrice $\mathcal{A}$ et sur chacune des colonnes de celle-ci nous ajouterons un 1. Sur cette ligne, nous mettrons ensuite $(q^3 - q^2) 0$. La matrice aura la forme suivante :

$$(3) \quad \mathcal{M} = \begin{array}{c|c} \mathcal{A} & e \\ \hline 11\dots 1 & 00\dots 0 \end{array}.$$

La matrice $\mathcal{C}$ à construire devra être de type $q^2 \times (q^3 - q^2)$ avec $N(\mathcal{C}) = q^2(q^2 - 1)$, telle que chaque ligne ait $q^2 - 1$ unités, chaque colonne $q + 1$ unités; deux lignes données ne sont coupées en des éléments unités que par q colonnes et trois lignes ne sont coupées en des unités que par une colonne. La construction d'une telle matrice n'apparaît pas immédiate,

Supposons donc connaître une matrice $\mathcal{A}$ à q^2 lignes et $q^2 + q$ colonnes avec $N = \sum_{i,k} a_{ik} = q^2(q+1)$, $\sum_k a_{ik} = q+1$, $\sum_k a_{ik} a_{jk} = 1$ pour tout $i \neq j$ et une matrice $\mathcal{C}$ à q^2 lignes et $q^3 - q^2$ colonnes $N = \sum_{i,m} c_{im} = (q+1)(q^3 - q^2)$, $\sum_i c_{im} = q+1$, $\sum_i c_{im} c_{in} = q$ pour tout $m \neq n$.

Formons la matrice (3); ses éléments seront

$$\begin{array}{llll} i = 1, 2, 3 & q^2 & m_{ij} = a_{ij} & \text{si } j \leq q^2 + q, \\ & & m_{ij} = c_{ij} & \text{si } j > q^2 + q, \\ i = q^2 + 1 & & m_{ij} = 1 & \text{si } j \leq q^2 + q, \\ & & m_{ij} = 0 & \text{si } j > q^2 + q. \end{array}$$

On aura $\sum_{i,j} m_{ij} = q(q+1) + q^2(q+1) + (q^3 - q^2)(q+1) = q(q+1)(q^2 + 1)$.

Pour appliquer le résultat de la Signorina Biscarini, il suffit de vérifier que pour tout $i \neq j$, $\sum_k m_{ik} m_{jk} = q+1$.

Si $j = q^2 + 1$, $\sum_k m_{ik} m_{jk} = \sum_k a_{ik} = q+1$ et si $j \neq q^2 + 1$, $\sum_k m_{ik} m_{jk} = \sum_k a_{ik} a_{jk} + \sum_k c_{ik} c_{jk} = q+1$.

$\mathcal{M}$ représente donc bien un plan de Möbius.

5. Si nous partons d'un champ de Galois avec $q = \pi^e$ où π est premier et e entier positif, nous connaissons un plan de Möbius $\mathcal{M}$ auquel par imposition d'un point correspond le plan affine $\mathcal{A}$, dont la matrice d'incidence et sa résiduelle $\mathcal{C}$ sont connues. S'il existe un autre plan affine non isomorphe $\mathcal{A}'$ la combinaison précédente sur $\mathcal{A}'$ et $\mathcal{C}$ permet de construire la matrice $\mathcal{M}'$ définissant un plan de Möbius. Reste à savoir s'il est ou non isomorphe à celui de départ.

Si $e = 1$, tous les points de l'ovoïde (5) auquel on peut rattacher le plan de Möbius sont équivalents et dans ce cas quelque soit le point projetant choisi on a la même structure pour $\mathcal{A}$; s'il existe un autre plan affine, ce qui reste non connu, le plan $\mathcal{M}'$ ainsi obtenu serait distinct de $\mathcal{M}$.

Dans le cas $e = 2$, il est possible que les points de l'ovoïde se distinguent en ceux appartenant au champ de caractéristique 3 et les autres, d'où peut-être deux structures différentes, mais par exemple si $q = 9$, on a quatre plans affines distincts; il est vraisemblable que par ce procédé on puisse obtenir un plan de Möbius différent de celui de départ.

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PLANE AFINE ȘI PLANE ALE LUI MÖBIUS

(Rezumat)

Se indică un procedeu menit eventual să contribuie la construcția planelor lui Möbius plecând de la plane deja cunoscute precum și de la plane afine.

У.Д.К. 517.512

О СОБСТВЕННЫХ ВЕКТОРАХ ТРИГОНОМЕТРИЧЕСКИХ МНОГОЧЛЕНОВ

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1. Матричная запись тр. (тригонометрических) рядов дает возможность представить алгебраические действия над рядами в виде преобразований аффинного пространства [2]...[5]. При этом возникает естественный вопрос о существовании собственных векторов, и в первой очереди, для тр. многочленов.

В работе решается

З а д а ч а. Для заданного тр. многочлена $f_n(x)$

$$(1) \quad f_n(x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx),$$

найти тр. многочлен $\varphi_n(x)$

$$(2) \quad \varphi_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^n (\alpha_k \cos kx + \beta_k \sin kx),$$

также имеющий $2n+1$ членов, чтобы

$$(3) \quad f_n(x) \varphi_n(x) = \lambda \varphi_n(x) + \psi_{2n}(x),$$

где λ — коэффициент пропорциональности и

$$(4) \quad \psi_{2n}(x) = \sum_{k=n+1}^{2n} (\gamma_k \cos kx + \delta_k \sin kx).$$

В основу решения положен матричный алгоритм умножения тр. рядов.

Если

$$(5) \quad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

$$(6) \quad \varphi(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} (\alpha_k \cos kx + \beta_k \sin kx),$$

есть тр. ряды функций интегрируемых с их квадратом, то коэффициенты произведения этих рядов

$$(7) \quad f(x) \varphi(x) = \frac{\gamma_0}{2} + \sum_{k=1}^{\infty} (\gamma_k \cos kx + \delta_k \sin kx),$$

можно получить линейно преобразуя коэффициенты одного из умножаемых рядов, т.е.

$$(8) \quad \|C\| = \|A\| [P(B)] = \|B\| [P(A)],$$

где

$$(9) \quad \|C\| = \left\| \dots \gamma_k, \dots \gamma_2, \gamma_1, \frac{\gamma_0}{2}, \delta_1, \delta_2, \dots \delta_k, \dots \right\|,$$

$$(10) \quad \|A\| = \left\| \dots a_k, \dots a_2, a_1, \frac{a_0}{2}, b_1, b_2, \dots b_k, \dots \right\|,$$

$$(11) \quad \|B\| = \left\| \dots \alpha_k, \dots \alpha_2, \alpha_1, \frac{\alpha_0}{2}, \beta_1, \beta_2, \dots \beta_k, \dots \right\|,$$

$[P(B)]$ и $[P(A)]$ — бесконечные матрицы с четырьмя входами одинаковой структуры. Первая составлена из координат вектора $\|B\|$, вторая из $\|A\|$

$$[P(A)] =$$

$$(12) = \frac{1}{2} \begin{array}{c|c|c|c|c|c|c|c|c|c} \dots a_0 + a_8 & a_1 + a_7 & a_2 + a_6 & a_3 + a_5 & a_4 & b_5 - b_3 & b_6 - b_2 & b_7 - b_1 & b_8 & \dots \\ \dots a_1 + a_7 & a_0 + a_6 & a_1 + a_5 & a_2 + a_4 & a_3 & b_4 - b_2 & b_5 - b_1 & b_5 & b_7 + b_1 & \dots \\ \dots a_2 + a_6 & a_1 + a_5 & a_0 + a_4 & a_1 + a_3 & a_2 & b_3 - b_1 & b_4 & b_5 + b_1 & b_6 + b_2 & \dots \\ \dots a_3 + a_5 & a_2 + a_4 & a_1 + a_3 & a_0 + a_2 & a_1 & b_2 & b_3 + b_1 & b_4 + b_2 & b_5 + b_3 & \dots \\ \hline \dots 2a_4 & 2a_3 & 2a_2 & 2a_1 & a_0 & 2b_1 & 2b_2 & 2b_3 & 2b_4 & \dots \\ \hline \dots b_5 - b_3 & b_4 - b_2 & b_3 - b_1 & b_2 & b_1 & a_0 - a_2 & a_1 - a_3 & a_2 - a_4 & a_3 - a_5 & \dots \\ \dots b_6 - b_2 & b_5 - b_1 & b_4 & b_3 + b_1 & b_2 & a_1 - a_3 & a_0 - a_4 & a_1 - a_5 & a_2 - a_6 & \dots \\ \dots b_7 - b_1 & b_6 & b_5 + b_1 & b_4 + b_2 & b_3 & a_2 - a_4 & a_1 - a_5 & a_0 - a_6 & a_1 - a_7 & \dots \\ \dots b_8 & b_7 + b_1 & b_6 + b_2 & b_5 + b_3 & b_4 & a_3 - a_5 & a_2 - a_6 & a_1 - a_7 & a_0 - a_8 & \dots \end{array}$$

Строковые бесконечные матрицы (9), (10) и (11), каждая с двумя входами, названы векторами коэффициентов тр. рядов.

Будем пользоваться нижеследующими векторами коэффициентов тр. многочленов:

$\|A_n\|$ и $\|B_n\|$ — многочленов $f_n(x)$ и $\varphi_n(x)$, которые есть строковые матрицы порядка $2n+1$. Например:

$$(13) \quad \|A_n\| = \left\| a_n, \dots, a_2, a_1, \frac{a_0}{2}, b_1, b_2, \dots, b_n \right\|;$$

$\|C_{2n}\|$ — тр. многочлена $f_n(x) \varphi_n(x)$, порядка $4n+1$,

$$(14) \quad \|C_{2n}\| = \left\| \gamma_{2n}, \dots, \gamma_2, \gamma_1, \frac{\gamma_0}{2}, \delta_1, \delta_2, \dots, \delta_{2n} \right\|.$$

Из (14), вычеркивая n первых и n последних элементов, получаем

$$(15) \quad \|C_n\| = \left\| \gamma_n, \dots, \gamma_2, \gamma_1, \frac{\gamma_0}{2}, \delta_1, \delta_2, \dots, \delta_n \right\|$$

строковую матрицу порядка $2n+1$.

$\|\tilde{C}\|$ — строковая матрица, порядка $2n$, составлена из n первых и n последних элементов матрицы (14),

$$(16) \quad \|\tilde{C}\| = \|\gamma_{2n} \dots \gamma_{n+2}, \gamma_{n+1}, \delta_{n+1}, \delta_{n+2} \dots \delta_{2n}\|.$$

2. Ответ на поставленную задачу дает

Теорема. Произведение заданного тр. многочлена $f_n(x)$, вектор коэффициентов которого $\|A_n\|$, на собственный вектор $\|B_n\|$ квадратной матрицы $[P(A_n)]_{n \times n}$, порядка $2n+1$

$$[P(A_n)]_{n \times n} =$$

$$(17) = \frac{1}{2} \left[\begin{array}{cccc|cccc} a_0 & a_1 \dots & a_{n-2} & a_{n-1} & a_2 & -b_{n-1} & -b_{n-2} \dots & -b_1 & 0 \\ \hline a_{n-2} & a_{n-3} \dots & a_0 + a_4 & a_1 + a_3 & a_2 & b_3 - b_1 & b_4 \dots & b_{n-3} & b_{n-2} \\ a_{n-1} & a_n + a_{n-2} \dots & a_1 + a_3 & a_0 + a_2 & a_1 & b_2 & b_3 + b_1 \dots & b_{n-2} + b_n & b_{n-1} \\ \hline 2a_n & 2a_{n-1} \dots & 2a_2 & 2a_1 & a_0 & 2b_1 & 2b_2 \dots & 2b_{n-1} & 2b_n \\ \hline -b_{n-1} & b_n - b_{n-2} \dots & b_3 - b_1 & b_2 & b_1 & a_0 - a_2 & a_1 - a_3 \dots & a_{n-2} - a_n & a_{n-1} \\ -b_{n-2} & -b_{n-3} \dots & b_4 & b_3 + b_1 & b_2 & a_1 - a_3 & a_0 - a_4 \dots & a_{n-3} & a_{n-2} \\ \hline 0 & b_1 \dots & b_{n-2} & b_{n-1} & b_n & a_{n-1} & a_{n-2} \dots & a_1 & a_0 \end{array} \right].$$

все элементы которой расположенные вне ромба, состоят из одного слагаемого, а остальные из двух слагаемых, есть тр. многочлен, первые $2n+1$ коэффициентов которого пропорциональны соответствующим коэффициентам собственного вектора, т.е.

$$(18) \quad \|C_n\| = \lambda \|B_n\|.$$

Коэффициент пропорциональности λ есть характеристическое или собственное число матрицы (17), причем все значения λ , полученные из характеристического уравнения являются действительными числами.

Остальные коэффициенты произведения (коэффициенты многочлена $\psi_{2n}(x)$) определяются равенством

$$(19) \quad \|\tilde{C}\| = \|B_n\| [P],$$

где $[P]$ — прямоугольная матрица порядка $(2n+1) \times 2n$,

$$(20) \quad [P] = \frac{1}{2} \begin{bmatrix} a_n \dots a_3 & a_2 & a_1 & b_1 & b_2 & b_3 \dots b_n \\ \hline 0 \dots a_n & a_{n-1} & a_{n-2} & b_{n-2} & b_{n-1} & b_n \dots 0 \\ 0 \dots 0 & a_n & a_{n-1} & b_{n-1} & b_n & 0 \dots 0 \\ 0 \dots 0 & 0 & a_n & b_n & 0 & 0 \dots 0 \\ 0 \dots 0 & 0 & 0 & 0 & 0 & 0 \dots 0 \\ 0 \dots 0 & 0 & -b_n & a_n & 0 & 0 \dots 0 \\ 0 \dots 0 & -b_n & -b_{n-1} & a_{n-1} & a_n & 0 \dots 0 \\ 0 \dots -b_n & -b_{n-1} & -b_{n-2} & a_{n-2} & a_{n-1} & a_n \dots 0 \\ \hline -b_n \dots -b_3 & -b_2 & -b_1 & a_1 & a_2 & a_3 \dots a_n \end{bmatrix}.$$

Доказательство. Матрица $[P(A_n)]$, составленная по (12) из координат заданного вектора $\|A_n\|$ (13), преобразует вектор коэффициентов тр. ряда в вектор коэффициентов произведения этого ряда на $\varphi(x)$. Если вместо ряда взять тр. многочлен $\varphi_n(x)$, то для получения вектора коэффициентов $\|C_{2n}\|$ произведения $f_n(x) \varphi_n(x)$ достаточно взять вырезку $[P(A_n)]_{n \times 2n}$ бесконечной матрицы $[P(A_n)]$, расположенную на n первых строк сверху и на n первых строк внизу от центральной и $2n$ первые левые и 2 первые правые столбцы от центрального. Таким образом полученную прямоугольную матрицу порядка $(2n+1) \times (4n+1)$ разделим вертикально на три клетки следующих порядков: $(2n+1) \times n$, $(2n+1) \times (2n+1)$ и $(2n+1) \times n$. Средняя клеточная матрица есть (17). Следовательно

$$(21) = \frac{1}{2} \begin{bmatrix} a_n \dots a_3 & a_2 & a_1 & & & \\ 0 \dots a_n & a_{n-1} & a_{n-2} & & & \\ 0 \dots 0 & a_n & a_{n-1} & & & \\ 0 \dots 0 & 0 & a_n & & & \\ 0 \dots 0 & 0 & 0 & & & \\ 0 \dots 0 & 0 & -b_n & & & \\ 0 \dots 0 & -b_n & -b_{n-1} & & & \\ 0 \dots -b_n & -b_{n-1} & -b_{n-2} & & & \\ -b_n \dots -b_3 & -b_2 & -b_1 & & & \end{bmatrix} [P(A_n)]_{n \times n} \begin{bmatrix} b_1 & b_2 & b_3 \dots b_n \\ b_{n-2} & b_{n-1} & b_n \dots 0 \\ b_{n-1} & b_n & 0 \dots 0 \\ b_n & 0 & 0 \dots 0 \\ 0 & 0 & 0 \dots 0 \\ a_n & 0 & 0 \dots 0 \\ a_{n-1} & a_n & 0 \dots 0 \\ a_{n-2} & a_{n-1} & 0 \dots 0 \\ a_1 & a_2 & a_3 \dots a_n \end{bmatrix}.$$

По (8) получаем вектор $\|C_{2n}\|$ произведения $f_n(x) \varphi_n(x)$

$$(22) \quad \|C_{2n}\| = \|B_n\| [P(A_n)]_{n \times 2n}.$$

Строковую матрицу $\|C_{2n}\|$ (14) разделяем также на три клетки, каждая из которых состоит из n , $2n+1$ и n элементов. Очевидно, что средняя клетка есть $\|C_n\|$ (15).

Применяя правила умножения клеточных матриц и соединяя в одно крайние клетки, получаем (19) и

$$(23) \quad \|C_n\| = \|B_n\| [P(A_n)]_{n \times n}.$$

Для выполнения условий (18) необходимо и достаточно:

$$(24) \quad \|B_n\| [P(A_n)]_{n \times n} = \lambda \|B_n\|,$$

или $\|B_n\|$ должно быть собственным вектором, а λ — собственным числом матрицы.

При записи $2n+1$ скалярных уравнений для отыскания $2n+1$ координат собственного вектора, все элементы центральной строки матрицы (17) делим на два, а зато умножаем на два центральный элемент собственного вектора $\|B_n\|$. Тогда получается характеристическое уравнение следующего вида

(25)

$$\begin{array}{cccccccccc|c}
 a_0 - \lambda & a_1 \dots & a_{n-2} & a_{n-1} & a_n & -b_{n-1} & -b_{n-2} \dots & -b_1 & 0 & \\
 \hline
 a_{n-2} & a_{n-3} \dots & a_0 + a_4 - \lambda & a_1 + a_3 & a_2 & b_3 - b_1 & b_4 \dots & b_{n-3} & b_{n-2} & \\
 a_{n-1} & a_n + a_{n-2} \dots & a_1 + a_3 & a_0 + a_2 - \lambda & a_1 & b_2 & b_1 + b_3 \dots & b_{n-2} + b_n & b_{n-1} & \\
 a_n & a_{n-1} \dots & a_2 & a_1 & \frac{a_0 - \lambda}{2} & b_1 & b_2 \dots & b_{n-1} & b_n & = 0. \\
 -b_{n-1} & b_n - b_{n-2} \dots & b_3 - b_1 & b_2 & b_1 & a_0 - a_2 - \lambda & a_1 - a_3 \dots & a_{n-2} - a_n & a_{n-1} & \\
 -b_{n-2} & -b_{n-3} \dots & b_4 & b_3 + b_1 & b_2 & a_1 - a_3 & a_0 - a_4 - \lambda \dots & a_{n-3} & a_{n-2} & \\
 \hline
 0 & b_1 \dots & b_{n-2} & b_{n-1} & b_n & a_{n-1} & a_{n-2} & a_1 & a_0 - \lambda &
 \end{array}$$

Определитель симметрический и все корни действительные числа.

Поступила 12 февраля 1971.

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ASUPRA VECTORILOR PROPRII AI POLINOAMELOR TRIGONOMETRICE

(Rezumat)

Continuîndu-și studiile referitoare la metoda matricial-armonică pe care au elaborat-o și care și-a găsit numeroase aplicații [6], autorii cercetează problema valorilor proprii ale polinoamelor trigonometrice.

EINE BEMERKUNG ZUR DIFFERENTIALGEOMETRIE IM QUASIELLIPTISCHEN RAUM

VON

PETER MEYER

Grünwald [1] gelangt zu der *Kinenatischen Abbildung* in der Weise, daß er die geordneten Punktpaare als Elemente der ebenen Geometrie zunächst in einem projektiven Raum P_5 mittels sechs homogener Koordinaten darstellt, die durch eine homogene, quadratische Relation miteinander verbunden sind. Die durch diese Gleichung beschriebene quadratische Mannigfaltigkeit von Punktpaaren der Ebene wird dann ausnahmslos eineindeutig auf die Plückerschen Geraden des dreidimensionalen Raumes im projektiven Sinn abgebildet. Im Gegensatz zu Grünwald stützt sich Blaschke [2], [3], [4] bei der Behandlung der „Kinematischen Abbildung“ auf eine Darstellung der ebenen Bewegungen und Umlegungen durch *Hamiltonsche Quaternionen* von Study [5]. Um eine einheitliche Beschreibung beider Transformationen zu erzielen, fügen Study wie auch Blaschke den vier (Hamiltonschen) Quaternioneneinheiten $e_0^*, e_1^*, e_2^*, e_3^*$ in bekannter Weise¹⁾ eine weitere Einheit ε mit der Rechenvorschrift $\varepsilon^2 = 0$ hinzu. Diesem Vorgehen entspricht der Grenzübergang von den Bewegungen im Strahlenbündel zu jenen der Ebene.

Betreibt man in dem geschlitzten *Parameter- oder Gruppenraum* der ebenen Bewegungen quasielliptische Differentialgeometrie, so zeigt sich beim Umgang mit diesen Quaternionen insofern eine kleine Inkonsistenz, als die Hilfseinheit ε zwar in Ansatz zu bringen ist, ε^2 als Faktor jedoch nicht in jedem Fall gleich Null gesetzt werden darf, vielmehr an einigen Stellen unberücksichtigt zu bleiben hat. Dies tritt ein in der quasielliptischen Kurventheorie bei der Normung und Festlegung des begleitenden Polartetraeders, der Einführung von Quasikrümmung und Quasiwindung sowie an entsprechender Stelle in der quasielliptischen Flächentheorie.

1) Vgl. S. 560 in [5].

Wie bereits aus oben zitierter Studyschen Untersuchung²⁾ hervorgeht, läßt sich die Einführung von ε folgendermaßen vermeiden: Von Strubecker [6] rührt eine direkte (d.h. nicht durch Grenzübergang gewonnene) Darstellung der ebenen Bewegungen durch Studysche (ausgeartete) Quaternionen e_0, e_1, e_2, e_3 her. Durch Adjungieren von vier weiteren Einheiten $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ zu den Quaternionen e_0, e_1, e_2, e_3 gelangt man im Sinne von Scheffers [7] zu einem Quaternionensystem aus acht Haupteinheiten³⁾, das nachfolgender Produkttafel gehorcht und ein assoziatives Zahlensystem bildet.

	e_0	e_1	e_2	e_3	ε_0	ε_1	ε_2	ε_3
e_0	e_0	e_1	e_2	e_3	ε_0	ε_1	ε_2	ε_3
e_1	e_1	$-e_0$	e_3	$-e_2$	ε_1	$-\varepsilon_0$	ε_3	$-\varepsilon_2$
e_2	e_2	$-e_3$	0	0	0	0	$-\varepsilon_0$	ε_1
e_3	e_3	e_2	0	0	0	0	$-\varepsilon_1$	$-\varepsilon_0$
ε_0	ε_0	ε_1	0	0	0	0	e_2	e_3
ε_1	ε_1	$-\varepsilon_0$	0	0	0	0	e_3	$-e_2$
ε_2	ε_2	$-\varepsilon_3$	$-\varepsilon_0$	ε_1	e_2	$-e_3$	$-e_0$	e_1
ε_3	ε_3	ε_2	$-\varepsilon_1$	$-\varepsilon_0$	e_3	e_2	$-e_1$	$-e_0$

Einer (Studyschen) Quaternion $A = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ mit $N(A) = a_0^2 + a_1^2 = 1$ ist nach Study und Strubecker eine Bewegung der Ebene $X' = \bar{A} X A$ ⁴⁾ zugeordnet. Einer (speziellen) Biquaternion (Begriffsbildung von Study) der Form $U = u_0 \varepsilon_0 + u_1 \varepsilon_1 + u_2 \varepsilon_2 + u_3 \varepsilon_3$ mit $N(U) = u_2^2 + u_3^2 = 1$ entspricht einer Umlegung der Ebene $X' = -\bar{U} X U$. Wie man leicht einsieht, bewahren alle bekannten Sätze über das Hintereinanderausüben von Bewegungen und Umlegungen bei dieser Schreibweise in acht Einheiten ihre Gültigkeit: z. B. führt die Zusammensetzung zweier Umlegungen U_1, U_2 zu einer Bewegung $A = U_1 U_2$ oder läßt sich eine

2) Vgl. § 2 und § 11 in [5].

3) Das Quaternionensystem aus acht Einheiten wurde von W. K. Clifford (Mathematical Papers, London 1882) aufgefunden, von Scheffers [7] in Verbindung mit räumlichen Bewegungen untersucht und von Study [5] zur Darstellung räumlicher sowie ebener Bewegungen und Umlegungen herangezogen.

4) Es sind a_i und im folgenden u_i ($i=0, 1, 2, 3$) reelle Zahlen bzw. Funktionen, $N(A)$ die Norm und $\bar{A}$ die konjugierte Quaternion von A , $X = x_1 e_1 + x_2 e_2 + x_3 e_3$, $X' = x'_1 e_1 + x'_2 e_2 + x'_3 e_3$. Darstellung des Punktes X durch Studysche Quaternionen in der Anfangs- bzw. gedrehten Lage.

Umlegung U immer in eine Bewegung A und eine darauffolgende Spiegelung S zerlegen: $U = AS$.

Unter Verwendung des Quaternionensystems $\{e_0, e_1, e_2, e_3, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ kann einerseits die „Kinematische Abbildung“ von Blaschke und Grünwald sachgerecht entwickelt werden, andererseits ist auch eine einwandfreie Behandlung der Differentialgeometrie des quasielliptischen Raumes gewährleistet. Von diesem Zahlensystem aus acht Einheiten kommt man sofort zu dem von Blaschke bevorzugten zurück: Jede Biquaternion $Q = \sum_{i=0}^3 (a_i e_i + u_i \varepsilon_i)$ gestattet vermöge der multiplikativen Verknüpfungsvorschrift die Darstellung $Q = (a_0 + \varepsilon_0 u_0)e_0 + (a_1 + \varepsilon_0 u_1)e_1 + (u_2 + \varepsilon_0 a_2)\varepsilon_2 + (u_3 + \varepsilon_0 a_3)\varepsilon_3$, in welcher $e_0, e_1, \varepsilon_2, \varepsilon_3$ als Grundeinheiten anzusehen sind und ε_0 mit ε zu identifizieren ist. Die Einheiten $e_0, e_1, \varepsilon_2, \varepsilon_3$ sind genau die Hamiltonschen Quaternionen (vgl. Multiplikationstafel).

Eingegangen am 25. Februar, 1970

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O OBSERVAȚIE PRIVIND GEOMETRIA DIFERENȚIALĂ ÎN SPAȚII CUASIELIPTICE

(Rezumat)

Utilizând un sistem cuaternionic cu opt unități se armonizează punctele de vedere ale lui Grünwald [1] și Blaschke [2]...[4], ceea ce permite o tratare nerestrictivă a geometriei diferențiale a spațiilor cuasieliptice.

CORRESPONDENȚA PRIN PARALELISM ÎNTRE VARIETĂȚI RIGLATE ȘI VARIETĂȚI CARACTERISTICE CORESPUNZĂTOARE ÎN S_5 AFIN ¹⁾

DE

ȘTEFANIA RUSCIOR

1. În această notă se consideră corespondența prin paralelism între varietăți riglate cu 3 dimensiuni din S_5 afin și varietățile caracteristice corespunzătoare. Am arătat [2], [3] că varietatea caracteristică se poate defini numai pe varietăți riglate cu trei dimensiuni de tip V_1^2 . În acest caz hiperplanele tangente duse de-a lungul unei generatoare generice g formează un fascicol de hiperplane cu baza într-un plan Π , în care este definită dreapta caracteristică d corespunzătoare generatoarei g [2]. Când g descrie varietatea V_3 , dreapta caracteristică d descrie tot o varietate riglată cu trei dimensiuni V_3^* numită varietate caracteristică. Varietatea dată V_3 are ecuația:

$$(1) \quad r(u, v) = \bar{\rho}(u, v) + w\bar{\alpha}(u, v),$$

unde $\bar{\rho} = \bar{\rho}(u, v)$ este o varietate directoare și $\bar{\alpha} = \bar{\alpha}(u, v)$ versorul generatoarei generice, definit în fiecare punct al varietății directoare. Funcțiile care definesc aceste elemente sînt din clasa C^1 , și satisfac condițiile analitice care caracterizează varietățile de tip (V_1^2) . Prin urmare avem [2]...[4]:

$$(2) \quad \bar{\alpha}_v = \lambda\bar{\alpha} + \mu\bar{\beta} + n\bar{\alpha}_u, \quad \bar{\rho}_v = l\bar{\alpha} + m\bar{\beta} + n\bar{\rho}_u.$$

Planul de bază Π este determinat de vectorii $\bar{\alpha}$ și $\bar{\beta}$ iar λ, μ, l, m și n sînt scalari ce depind de u și v .

Varietatea caracteristică V_3^* are ecuația:

¹⁾ Prezentată la sesiunea științifică jubiliară „Al. Myller“ a Universității „Al. I. Cuza“ din Iași, din 20-25 august 1970.

$$(3) \quad \bar{r}^* = \bar{\rho}^*(u, v) + w^* \bar{\alpha}^*(u, v),$$

în care avem:

$$(4) \quad \begin{cases} \bar{\rho}^*(u, v) = \bar{\rho}(u, v) + a\bar{\alpha} + b\bar{\beta}, \\ \bar{\alpha}^*(u, v) = m\bar{\alpha} + n\bar{\beta} + p\bar{\rho}_u. \end{cases}$$

Calculînd pe α_v^* și ρ_u^* din (4) se constată că varietatea V_3^* este tot de tipul V_1^2 , a , b și p fiind scalari care depind de u și v . Conform unei teoreme demonstrate anterior [4], varietatea caracteristică V_3^* și varietatea V_3 fiind din aceeași clasă, pot fi puse în corespondență prin hiperplane tangente paralele cu păstrarea generatoarelor, generatoarele corespunzătoare fiind g și d . Deoarece hiperplanele tangente corespunzătoare sînt varietăți liniare cu trei dimensiuni H și H^* definite în fiecare punct nesingular în S_5 , ele sînt complet paralele sau parțial paralele de dimensiune $\delta = 2/3$. În adevăr, dacă introducem funcția numerică care caracterizează dimensiunea δ a paralelismului parțial [5] avem:

a) cazul cînd hiperplanele tangente H și H^* admit ca intersecție un plan impropriu, $\delta = 1$;

b) cazul cînd aceste hiperplane se intersectează după o dreaptă improprie, $\delta = 2/3$.

2. Pentru a stabili o corespondență prin hiperplane tangente complet paralele considerăm fascicolul de hiperplane H care are baza planul Π determinat de direcțiile $\bar{\alpha}$ și $\bar{\beta}$. Secțiunea acestui fascicol cu hiperplanul impropriu este un fascicol de plane P ce are ca suport dreapta Δ dată de punctele $\bar{\alpha}$, $\bar{\beta}$. Punctele generatoarei g se corespund proiectiv cu planele P . Secțiunea fascicolului H^* cu hiperplanul impropriu este un fascicol de plane P^* ce are ca suport dreapta Δ^* determinată de $\bar{\alpha}^*$ și $\bar{\beta}^*$. Punctele generatoarei g^* se corespund proiectiv cu planele P^* . Pentru a obține hiperplane tangente H^* complet paralele cu H impunem condiția ca dreapta improprie Δ^* să fie coplanară cu Δ . Această condiție este îndeplinită numai dacă planele de bază Π^* și Π sînt semiparalele adică au un punct impropriu I comun $I = \Delta \cap \Delta^*$, fapt care se exprimă analitic prin condiția ca rangul matricei

$$(5) \quad M = \|\bar{\alpha}, \bar{\beta}, \bar{\alpha}^*, \bar{\beta}^*\|$$

să fie trei. Dacă această condiție este îndeplinită, considerînd un punct $M^* \in g^*$, găsim $H = \Pi \cup M^*$ care admite ca secțiune improprie planul determinat de $\bar{\alpha}$, $\bar{\beta}$ și $\bar{\alpha}^*$. Hiperplanul H are un punct de tangență $M \in V_3$ și este complet paralel cu H^* dus în M^* , deoarece H^* admite ca secțiune improprie planul determinat de punctele $\bar{\alpha}^*$, $\bar{\beta}^*$ și $\bar{\alpha}$ care este identic cu planul determinat de punctele $\bar{\alpha}$, $\bar{\beta}$ și $\bar{\alpha}^*$. Impunînd condiția de paralelism complet al hiperplanelor H și H^* rezultă relația:

$$(6) \quad Aww^* + Bw + Cw^* = 0,$$

în care:

$$\begin{cases} A = [\alpha, \bar{\beta}, \bar{\alpha}_u, \bar{\beta}_u, \bar{\rho}_u^*] + [\bar{\alpha}, \bar{\beta}, \bar{\alpha}_v, \bar{\beta}_v, \bar{\rho}_v^*], \\ B = [\bar{\alpha}_u, \bar{\beta}_u, \bar{\rho}_v, \bar{\alpha}_v, \bar{\alpha}] + [\bar{\alpha}_u, \bar{\beta}_u, \bar{\rho}_u, \bar{\alpha}_u, \bar{\alpha}], \\ C = [\bar{\alpha}_u, \bar{\beta}_u, \bar{\rho}_v, \bar{\alpha}_v^*, \bar{\alpha}] + [\bar{\alpha}_u, \bar{\beta}_u, \bar{\rho}_u, \bar{\alpha}_u^*, \bar{\alpha}^*]. \end{cases}$$

Obținem astfel următoarele rezultate:

1. Condiția necesară și suficientă ca să stabilim o corespondență prin hiperplane tangente complet paralele între o varietate V_3 și varietatea caracteristică V_3^* este ca planele de bază ale generatoarelor corespunzătoare să fie semiparalele.

2. Generatoarele corespunzătoare sînt generatoarea g și dreapta ei caracteristică.

3. Corespondența între punctele generatoarelor corespunzătoare este o omologie (6).

Se constată că pentru fiecare pereche de generatoare corespunzătoare există un centru de omologie. Atunci cînd g descrie varietatea V_3 , centrul de omologie descrie o dreaptă ce admite ca punct impropriu punctul $I = \Delta \cap \Delta^*$.

3. În cazul cînd hiperplanele H și H^* sînt parțial paralele de dimensiune $\delta = 2/3$ ele se intersectează după o dreaptă improprie. A fost studiată problema în ipoteza $\alpha = \alpha^*$. Hiperplanele tangente H și H^* au un punct impropriu comun α : pentru a avea o dreaptă comună este suficient ca matricea:

$$M_1 = \|r_u, r_v, \alpha, r_u^*, r_v^*\|$$

să aibă rangul trei.

Exprimînd condițiile într-un sistem local convenabil ales, format din vectorii $\alpha, \bar{\beta}, \bar{\alpha}_u, \bar{\beta}_u, \bar{\rho}_u$ se obține relația:

$$(7) \quad (Aw + Bw^* + C)(A_1w + B_1w^* + C_1) = 0,$$

care determină punctele corespunzătoare în cazul corespondenței impuse. Coeficienții A, B, C, A_1, B_1, C_1 , sînt determinați în funcție de elementele varietății date. Obținem următorul rezultat:

Corespondența prin hiperplane tangente parțial paralele, de dimensiune $\delta = 2/3$, considerată între o varietate riglată V_3 și varietatea caracteristică corespunzătoare V_3^* pentru cazul cînd generatoarele corespunzătoare sînt paralele este un produs de două omotetii.

Primită la 22 martie 1971

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LA CORRESPONDANCE PAR PARALLÉLISME ENTRE VARIÉTÉS RÉGLÉES
ET VARIÉTÉS CARACTÉRISTIQUES DANS S_5 AFFIN

(Résumé)

On considère une correspondance par variétés linéaires tangentes entre une variété réglée de dimension 3 et sa variété caractéristique. Dans S_5 affin les variétés linéaires peuvent être soit complètement parallèles ($\delta = 1$) ou bien partiellement parallèles ($\delta = 2/3$). Dans cette Note on étudie les correspondances pour ces deux cas entre les variétés réglées qui conservent leur génératrices. On obtient les résultats suivants :

1. Les correspondances introduites peuvent être obtenues seulement pour les variétés réglées qui admettent des variétés caractéristiques du même type, dont les génératrices correspondantes sont situées dans deux plans de base semi-parallèles ($\delta = 1/2$).
2. La correspondance par parallélisme complet détermine entre les points des génératrices correspondantes une homologie.
3. La correspondance par parallélisme partiel, établie dans le cas des génératrices correspondantes parallèles, est constituée par deux homothéties.

CONTINUOUS FAMILIES OF TOPOLOGICAL SUBSPACES

BY

J. B. WILKER

1. Definition and Characterization

A family A_i ($i \in I$) of subspaces of a topological space X is called a *continuous family* if every function continuous on each of the subspaces is necessarily continuous on their union. Such families are characterized in the first theorem.

Theorem 1. *Let A_i ($i \in I$) be a family of subspaces of the space X . Then the following conditions are equivalent:*

(a) *If Y is any topological space and f any Y -valued function on X with the property that the restrictions $f|A_i$ ($i \in I$) are all continuous, then $f| \bigcup_{i \in I} A_i$ is also continuous [i.e. A_i ($i \in I$) is a continuous family of subspaces].*

(b) *Let Y_0 be the set $\bigcup_{i \in I} A_i$ endowed with the topology in which a set is open if and only if it meets each A_i in an open set. Then condition (a) holds for $Y = Y_0$.*

(c) *The set $\bigcup_{i \in I} A_i$ inherits the topology described in (b) as a subspace of X .*

Proof. Condition (a) implies the special case in condition (b).

Any Y_0 -valued function which restricts to the identity on $\bigcup_{i \in I} A_i$ is open on $\bigcup_{i \in I} A_i$. But it is also continuous on each A_i so that in the presence of condition (b) it is a homeomorphism on $\bigcup_{i \in I} A_i$ and condition (c) follows.

If $f: X \rightarrow Y$ is any function and O an open subset of Y then

$$(f| \bigcup_{i \in I} A_i)^{-1}(O) = \bigcup_{i \in I} (f|A_i)^{-1}(O).$$

By condition (c) the continuity of f on each A_i implies its continuity on $\bigcup_{i \in I} A_i$ and we have condition (a).

2. Continuous Pairs

For continuous families consisting of two subspaces it is possible to replace condition (c) with a characterization which does not depend on relative topology. Well known sufficient conditions of this type include the following.

If two subspaces A and B are both open or both closed, they form a continuous pair. The second condition is used for example in proving the transitivity of the homotopy equivalence of curves ([2, p. 24, Theorem 5]).

J. L. Kelley ([1, p. 53, Corollary 19 and p. 100, Problem B]) offers the more general sufficient condition that the sets $A - B$ and $B - A$ be separated. The subspaces $A = \{1, 2\}$ and $B = \{2, 3\}$ of the topological space $X = \{1, 2, 3\}$ with the indiscrete topology give a very simple example of a continuous pair which does not satisfy this condition. However, the characterization of continuous pairs given in Theorem 2 shows that if the subspaces are disjoint, then Kelley's condition, $\bar{A} \cap B = \bar{B} \cap A = \emptyset$, is necessary.

Theorem 2. *Let A and B be subspaces of a topological space X . Then A and B form a continuous pair if and only if for any subset S of $A - B$, $\bar{S} \cap B = \overline{S \cap A \cap B} \cap B$ and, symmetrically, for any subset S of $B - A$, $\bar{S} \cap A = \overline{S \cap B \cap A} \cap A$.*

Remarks. In the language of relative topology the first half of the condition says that the closure in X of an arbitrary subset S of $A - B$ meets B in the closure in B of that part of its closure in A which overlaps B .

It is tempting to conjecture that Theorem 2 could be simplified by testing only $S = A - B$ and $B - A$ respectively. However the plane subspaces $A = \{(O, y) : 0 \leq y \leq 1\}$ and $B = \{(x, y) : 0 < x^2 + y^2 \leq 1\}$ satisfy this oversimplified condition but do not form a continuous pair.

Proof. The condition in Theorem 1 (c) is equivalent to saying that a subset of $A \cup B$ is closed if and only if it meets A and B in closed sets.

Suppose A and B form a continuous pair and $S \subset A - B$. As $\bar{S} \supset \bar{S} \cap A \cap B$, $\bar{S} \cap B \supset \overline{S \cap A \cap B} \cap B$. To work for the reverse inclusion consider $T = (\bar{S} \cap A) \cup (\overline{S \cap A \cap B})$. T is closed in $A \cup B$ because $T \cap A = \bar{S} \cap A$ which is closed in A and $T \cap B = \overline{S \cap A \cap B} \cap B$ which is closed in B . But $T \supset \bar{S} \cap A \supset S$ and so $T \supset \bar{S} \cap (A \cup B)$, the closure of S in $A \cup B$. That is

$$(\bar{S} \cap A) \cup (\overline{S \cap A \cap B}) \supset (\bar{S} \cap A) \cup (\bar{S} \cap B).$$

The trace of this in B simplifies to the required

$$\bar{S} \cap A \cap B \cap B \supset \bar{S} \cap B.$$

Let us use the conditions of the Theorem to show that if $S \subset A \cup B$ meets A and B in closed sets then it is closed in $A \cup B$. That is, if $\overline{S \cap A} \cap A = S \cap A$ and $\overline{S \cap B} \cap B = S \cap B$ then $\overline{S \cap (A \cup B)} = S \cap (A \cup B)$. Such as S may be written as

$$S = [S \cap (A - B)] \cup [S \cap A \cap B] \cup [S \cap (B - A)] = S_1 \cup S_2 \cup S_3.$$

It is sufficient to show that $\overline{S_i \cap (A \cup B)} \subset S \cap (A \cup B)$ ($i = 1, 2, 3$). As $S_2 \subset S \cap A$, $\overline{S_2 \cap A} \subset \overline{S \cap A} \cap A = S \cap A$. Similarly $\overline{S_2 \cap B} \subset S \cap B$ and therefore

$$\overline{S_2 \cap (A \cup B)} = (\overline{S_2 \cap A}) \cup (\overline{S_2 \cap B}) \subset (S \cap A) \cup (S \cap B) = S \cap (A \cup B).$$

The arguments for S_1 and S_3 are symmetric. As $S_1 \subset S \cap A$, $\overline{S_1 \cap A} \subset S \cap A$. The condition of the Theorem and this inclusion show that $\overline{S_1 \cap B} = \overline{S_1 \cap A \cap B \cap B} \subset \overline{S \cap A \cap B \cap B} \subset \overline{S \cap B \cap B} = S \cap B$.

It follows that $\overline{S_1 \cap (A \cup B)} \subset S \cap (A \cup B)$.

3. Continuous Pairs and Continuous Families

The existence of enough continuous pairs can prove a finite family is continuous. Two different ways in which this can happen are described in the following Theorem.

Theorem 3. *The finite family A_i ($i = 1, 2, \dots, n$) of subspaces of X is a continuous family if each subspace A_j forms a continuous pair with the union of its predecessors $\bigcup_{i=1}^{j-1} A_i$, or if every two of its members form a continuous pair.*

Proof. Suppose $f: X \rightarrow Y$ has continuous restrictions $f|A_i$ ($i = 1, 2, \dots, n$). In the first case successive applications of the definition of continuous pair give the continuity of $f| \bigcup_{i=1}^j A_i$ ($j = 2, 3, \dots, n$). In the second case it suffices to prove that $f| \bigcup_{i=1}^n A_i$ is continuous at an arbitrary point $a \in \bigcup_{i=1}^n A_i$.

Suppose for notational convenience that $a \in A_1$ and let N be a neighbourhood of $f(a)$ in Y .

Since A_1 and A_i ($i = 2, 3, \dots, n$) form a continuous pair there is a neighbourhood N_i of a in X such that $f(N_i \cap (A_1 \cup A_i)) \subset N$. Then

$N_0 = \bigcup_{i=2}^n N_i$ is a neighbourhood of a in X ,

$$f(N_0 \cap (\bigcup_{i=1}^n A_i)) \subset \bigcup_{i=2}^n f(N_i \cap (A_1 \cup A_i)) \subset N$$

and $f|_{\bigcup_{i=1}^n A_i}$ is continuous at a .

Neither of the conditions of Theorem 3 is sufficient for infinite families of subspaces. A countable counterexample is obtained by taking the A_i ($i = 1, 2, \dots$) to be the singletons from a convergent sequence of distinct points and its limit in a T_1 space.

Both of the conditions of Theorem 3 require at least two of the subspaces in the family to form a continuous pair. The following example of a continuous triple without continuous pairs shows that this condition is not necessary.

Let X be the complex plane. Let A_1 be the subset of the unit circle defined by

$$A_1 = \left\{ e^{i\theta} : \frac{\pi}{6} < \theta \leq \frac{5\pi}{6} \text{ or } \frac{5\pi}{4} < \theta < \frac{7\pi}{4} \right\}$$

and let A_2 and A_3 be the rotated sets $e^{2\pi i/3} A_1$ and $e^{4\pi i/3} A_1$ respectively. No two of these subspaces form a continuous pair because they have abutting open and closed ends. But the three of them form a continuous triple because each „smoothes“ the junction of the other two.

4. An Example Suggesting Further Research

The subspaces $A = \{x : 0 < x < 1\}$ and $B = \{x : x \text{ is rational and } 0 \leq x \leq 1\}$ of the real line do not form a continuous pair because $S = \{1 - \sqrt{2}/n : n = 2, 3, \dots\}$ is closed in A but not in $A \cup B$. Nevertheless any metric space-valued function f which is continuous on A and on B is continuous on $A \cup B$. To establish the continuity of f at 0 consider any sequence $x_i \rightarrow 0$ in A and choose a corresponding sequence $y_i \rightarrow 0$ in B such that $d(f(x_i), f(y_i)) < i^{-1}$. It then follows that $\lim_{i \rightarrow \infty} f(x_i) = \lim_{i \rightarrow \infty} f(y_i) = f(0)$. An analogous argument proves the continuity of f at 1.

The example suggests that we might profitably introduce the concept of continuous pair or continuous family with respect to a certain testing range or family of testing ranges. In this language the subspaces A and B of the example form a continuous pair with respect to metric spaces.

Two kinds of problems would then seem natural. First, characterize continuous pairs and families with respect to metric spaces, compact spaces, Hausdorff spaces, etc. Second, given a pair or family of topological subspaces, find the most general testing range with respect to which they form a continuous pair or family. Interesting characterizations of familiar classes of spaces might arise out of these proposed investigations.

Received September 12, 1970

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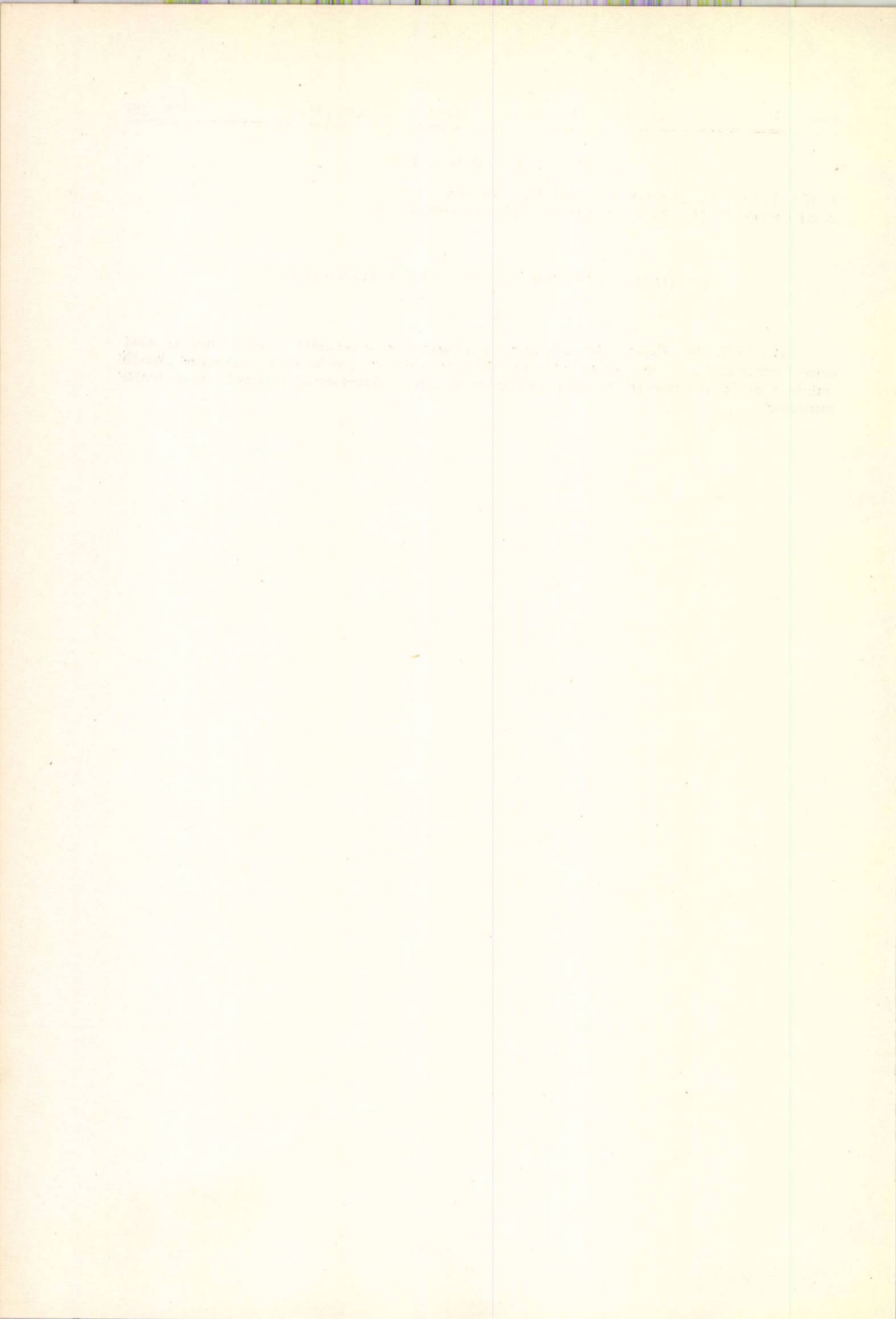
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FAMILII CONTINUE DE SUBSPAȚII TOPOLOGICE

(Rezumat)

O familie de subspații ale unui spațiu topologic dat se numește *familie continuă* dacă orice funcție continuă pe fiecare din subspații este necesar continuă pe reuniunea lor. Se stabilesc unele teoreme și se indică probleme ce pot fi cercetate în legătură cu noțiunile introduse.



FIBRĂRI ÎN CATEGORIA PERECHILOR $(C_p)^1$

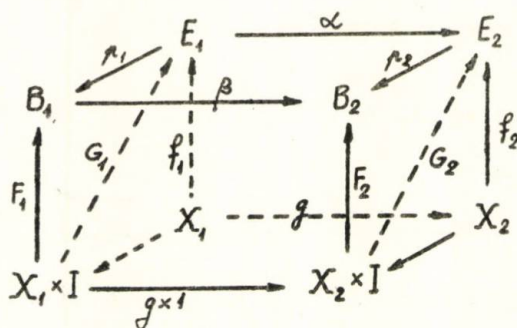
DE

I. POP

1. În această Notă se consideră fibrările în categoria perechilor C_p , se dau condiții necesare și suficiente ca un morfism în categoria C_p să fie o fibrare, folosind noțiunea de morfism de ridicare, și se stabilește legătura cu fibrările slabe [3]; de asemenea se introduce noțiunea de echivalență homotopică a două fibrări în C_p și se definește un functor contravariant care asociază unui obiect din C_p mulțimea claselor de fibrări în C_p echivalente homotopic și avînd obiectul dat ca bază.

2. Definiția 1. Un morfism $p = (p_1, p_2): \alpha \rightarrow \beta$ în categoria C_p are proprietatea de ridicare a homotopiei în raport cu un obiect $g: X_1 \rightarrow X_2$ dacă oricare ar fi un morfism $\gamma: g \rightarrow \alpha$ și oricare ar fi o homotopie Γ pentru $p\gamma$ să existe o homotopie Γ_1 pentru γ încît $p\Gamma_1 = \Gamma$.

Acastă definiție este explicitată în următoarea diagramă comutativă:



Definiția 2. Un morfism $p: \alpha \rightarrow \beta$ în categoria C_p este o fibrare Hurewicz dacă p are proprietatea de ridicare a homotopiei în raport cu orice obiect g din C_p .

¹⁾ Prezentată la sesiunea științifică a Institutului politehnic din Iași, din 2-4 iulie 1970.

Fie acum diagrama comutativă:

$$\begin{array}{ccc} E_1 & \xrightarrow{p_1} & B_1 \\ \alpha \downarrow & & \downarrow \beta \\ E_2 & \xrightarrow{p_2} & B_2 \end{array}$$

Să notăm

$$\text{Hur}(E_1) = \overline{E_1} \times B_1^I = \{(e_1, \lambda_1)/\lambda_1(0) = p_1(e_1)\} \text{ și}$$

$\text{Hur}(E_2) = \overline{E_2} \times B_2^I = \{(e_2, \lambda_2)/\lambda_2(0) = p_2(e_2)\}$ și să considerăm în C_p obiectul $\alpha \times \beta^I: \overline{E_1} \times B_1^I \longrightarrow \overline{E_2} \times B_2^I$ cu $(\alpha \times \beta^I)(e_1, \lambda_1) = (\alpha(e_1), \beta\lambda_1)$.

În adevăr din $\lambda_1(0) = p_1(e_1)$ rezultă $\beta\lambda_1(0) = \beta p_1(e_1) = p_2(\alpha(e_1))$.

Să mai considerăm obiectul $\alpha^I: E_1^I \longrightarrow E_2^I$ cu $\alpha^I(\mu) = \alpha\mu$.

Definiția 3. Dat un morfism $p = (p_1, p_2)$ în categoria C_p , vom numi morfism de ridicare pentru p un morfism $\Phi = (\Phi_1, \Phi_2): \alpha \times \beta^I \longrightarrow \alpha^I$ cu condițiile: 1° $\Phi_i(e_i, \lambda_i) = e_i$ și 2° $p_i \Phi_i(e_i, \lambda_i) = \lambda_i$, ($i = 1, 2$).

Teorema 1. Condiția necesară și suficientă pentru ca un morfism $p = (p_1, p_2): \alpha \longrightarrow \beta$ să fie o fibrare Hurewicz în categoria C_p este ca pentru p să existe un morfism de ridicare.

Demonstrație. Să dovedim mai întâi necesitatea. Fie pentru aceasta diagrama

$$\begin{array}{ccccc} & E_1 & \xrightarrow{\alpha} & E_2 & \\ & \nearrow p_1 & & \nwarrow p_2 & \\ B_1 & \xrightarrow{\beta} & B_2 & & \\ \uparrow f_1 & & \uparrow f_2 & & \\ \overline{E_1} \times B_1^I & \xrightarrow{\alpha \times \beta^I} & \overline{E_2} \times B_2^I & & \\ \uparrow F_1 & & \uparrow F_2 & & \\ \overline{E_1} \times B_1^I \times I & \xrightarrow{\alpha \times \beta^I \times 1} & \overline{E_2} \times B_2^I \times I & & \end{array}$$

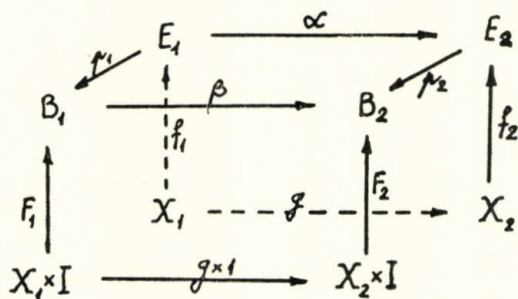
cu $f_1(e_1, \lambda) = e_1$; $f_2(e_2, \mu) = e_2$. $F_1((e_1, \lambda), t) = \lambda(t)$ și $F_2((e_2, \mu), t) = \mu(t)$. Avem atunci $\alpha f_1(e_1, \lambda) = \alpha(e_1)$, $f_2(\alpha \times \beta^I)(e_1, \lambda) = f_2(\alpha(e_1), \beta\lambda) = \alpha(e_1)$ deci $\alpha f_1 = f_2(\alpha \times \beta^I)$. În plus $F((e_1, \lambda), 0) = \lambda(0) = p_1 f_1(e_1, \lambda) = p_1(e_1)$ și $F_2((e_2, \mu), 0) = \mu(0) = p_2(e_2) = p_2 f_2(e_2, \mu)$. În sfârșit $\beta F_1((e_1, \lambda), t) = \beta\lambda(t)$ și $F_2(\alpha \times \beta^I \times 1)((e_1, \lambda), t) = F_2((\alpha(e_1), \beta\lambda), t) = \beta\lambda(t)$ deci $\beta F_1 = F_2(\alpha \times \beta^I \times 1)$.

Toate aceste condiții fiind îndeplinite și cum p a fost presupusă fibrare Hurewicz în C_p , rezultă că există $G_1: \overline{E_1} \times B_1^I \times I \longrightarrow E_1$ și

$G_2: E_2 \times B_2^I \times I \longrightarrow E_2$ încît $G_i((e_i, \lambda_i), 0) = f_i(e_i, \lambda_i) = e_i$; $p_i G_i((e_i, \lambda_i), t) = \lambda_i(t)$ și $\alpha G_1 = G_2(\alpha \times \beta^I \times 1)$.

Definim acum $\Phi_i: E_i \times B_i^I \longrightarrow E_i$ prin $\Phi_i(e_i, \lambda_i)(t) = G_i((e_i, \lambda_i), t)$, ($i = 1, 2$). Să arătăm că $\Phi = (\Phi_1, \Phi_2)$ este un morfism de ridicare pentru p . Avem $\Phi_i(e_i, \lambda_i)(0) = G_i((e_i, \lambda_i), 0) = e_i$; $p_i \Phi_i(e_i, \lambda_i)(t) = \lambda_i(t)$. În plus $\alpha^I \Phi_1(e_1, \lambda)(t) = \alpha \Phi_1(e_1, \lambda)(t) = \alpha G_1(e_1, \lambda)(t)$ și $\Phi_2(\alpha \times \beta^I)(e_1, \lambda)(t) = \Phi_2(\alpha(e_1), \beta \lambda) = G_2(\alpha(e_1), \beta \lambda), t)$.

Să dovedim acum suficiența; fie pentru aceasta diagrama



cu $p_i f_i(x_i) = F_i(x_i, 0)$, ($i = 1, 2$), $\alpha f_1 = f_2 g$ și $\beta F_1 = F_2(g \times 1)$. Pentru $x_i \in X_i$ definim un drum $F_i^{x_i} \in B_i^I$ cu $F_i^{x_i}(t) = F_i(x_i, t)$. Să observăm că avem $(F_i^{x_i}, f_i(x_i)) \in E_i \times B_i^I$. În adevăr $F_i^{x_i}(0) = F_i(x_i, 0) = p_i f_i(x_i)$.

Definim acum $G_i: X_i \times I \longrightarrow E_i$ prin $G_i(x_i, t) = \Phi_i(F_i^{x_i}, f_i(x_i))(t)$. Avem atunci $G_i(x_i, 0) = f_i(x_i)$; $p_i G_i(x_i, t) = F_i^{x_i}(t) = F_i(x_i, t)$. În sfârșit $\alpha G_1(x_1, t) = \alpha \Phi_1(F_1^{x_1}, f_1(x_1))(t) = \Phi_2(\beta F_1^{x_1}, \alpha f_1(x_1))(t) = \Phi_2(\beta F_1^{x_1}, f_2 g(x_1))(t) = \Phi_2(F_2^{g(x_1)}, f_2 g(x_1))(t) = G_2(g(x_1), t) = G_2(g \times 1)(x_1, t)$. Am folosit aici egalitatea $\beta F_1^{x_1}(t) = \beta F_1(x_1, t) \equiv F_2(g(x_1), t) = F_2^{g(x_1)}$. Avem deci $\alpha G_1 = G_2(g \times 1)$ și deci p este fibrare în C_p .

Observație. Dacă $\Phi = (\Phi_1, \Phi_2)$ este un morfism de ridicare pentru $p = (p_1, p_2): \alpha \longrightarrow \beta$ atunci Φ_1 este o funcție de ridicare pentru p_1 iar Φ_2 este o funcție de ridicare pentru p_2 .

Corolar. Dacă $p = (p_1, p_2)$ este o fibrare Hurewicz în categoria C_p atunci p este o fibrare slabă în C_p [3].

Un rezultat reciproc este dat sub forma următoarei propoziții.

Propoziția 2. Dacă $p = (p_1, p_2)$ este o fibrare slabă în C_p și p_2 are proprietatea de drum unic ridicat, atunci p este fibrare Hurewicz.

Demonstrație. Fie $\Phi_i: E_i \times B_i^I \longrightarrow E_i$ funcții de ridicare pentru p_i (presupuse fibrări Hurewicz în categoria Top). Să arătăm că în ipoteza de mai sus $\Phi = (\Phi_1, \Phi_2)$ este un morfism de ridicare pentru p . Este suficient să arătăm că $\alpha^I \Phi_1 = \Phi_2(\alpha \times \beta^I)$, adică $\alpha \Phi_1(e_1, \lambda) = \Phi_2(\alpha(e_1), \beta \lambda)$. Avem $\alpha \Phi_1(e_1, \lambda)(0) = \alpha(e_1)$ și $\Phi_2(\alpha(e_1), \beta \lambda)(0) = \alpha(e_1)$ și $p_2 \alpha \Phi_1(e_1, \lambda) = \beta p_1 \Phi_1(e_1, \lambda) = \beta \lambda = p_2 \Phi_2(\alpha(e_1), \beta \lambda)$ și cum p_2 are proprietatea de drum

unic ridicat rezultă $\alpha^I \Phi_1 = \Phi_2(\alpha \times \beta^I)$. După teorema 1 rezultă că p este fibrare Hurewicz în C_p , ceea ce demonstrează afirmația de mai sus.

Vom utiliza teorema 1 pentru a arăta că fiecărui obiect în categoria $(C_p)_p$ i se poate asocia în mod natural o fibrare Hurewicz în categoria C_p .

Fie un astfel de obiect (v. diagrama de la definiția 2); să considerăm subspațiile $\text{Hur}(E_i) \subset E_i \times B_i^I$ definite anterior și definim aplicațiile: $\text{Hur}(p_i): \text{Hur}(E_i) \longrightarrow B_i$ cu $\text{Hur}(p_i)(e_i, \lambda_i) = \lambda_i(1)$. Avem atunci diagrama comutativă

$$\begin{array}{ccc} \text{Hur}(E_1) & \xrightarrow{\text{Hur}(p_1)} & B_1 \\ \alpha \times \beta^I = \alpha' \downarrow & & \downarrow \beta \\ \text{Hur}(E_2) & \xrightarrow{\text{Hur}(p_2)} & B_2 \end{array}$$

În adevăr $\beta \text{Hur}(p_1)(e_1, \lambda_1) = \beta \lambda_1(1)$ și $\text{Hur}(p_2)\alpha'(e_1, \lambda_1) = \text{Hur}(p_2)\alpha((e_1, \beta \lambda_1)) = \beta \lambda_1(1)$.

Să arătăm că are loc

Propoziția 3. *Morfismul $\text{Hur}(p) = (\text{Hur}(p_1), \text{Hur}(p_2)): \alpha' \longrightarrow \beta$ este o fibrare Hurewicz în C_p .*

Demonstrație. Vom arăta că pentru $\text{Hur}(p)$ există un morfism de ridicare. Fie $((e_1, \lambda_1), \mu_1) \in \text{Hur}(E_1) \times B_1^I$ adică $\lambda_1(0) = p_1(e_1)$ și $\mu_1(0) = \lambda_1(1)$. Definim un drum $\mu_{1t}: I \longrightarrow B_1$ cu $\mu_{1t}(\tau) = \mu_1(t\tau)$. Avem $\mu_{1t}(0) = \mu_1(0) = \lambda_1(1)$ și deci putem considera $\mu_{1t} \circ \lambda_1 \in B_1^I$. Definim acum $\Phi_1: \text{Hur}(E_1) \times B_1^I \longrightarrow \text{Hur}(E_1)^I$ prin $\Phi_1((e_1, \lambda_1), \mu_1)(t) = (e_1, \mu_{1t} \circ \lambda_1)$. În adevăr $(\mu_{1t} \circ \lambda_1)(0) = \lambda_1(0) = p_1(e_1)$. Mai avem $\Phi_1((e_1, \lambda_1), \mu_1)(0) = (e_1, \mu_{10} \circ \lambda_1) = (e_1, \lambda_1)$ deoarece $\mu_{10}(\tau) = \mu_1(0) = \lambda_1(1)$. În sfârșit $\text{Hur}(p_1)\Phi_1((e_1, \lambda_1), \mu_1)(t) = (\mu_{1t} \circ \lambda_1)(1) = \mu_{1t}(1) = \mu_1(t)$, deci $\text{Hur}(p_1)\Phi_1((e_1, \lambda_1), \mu_1) = \mu_1$.

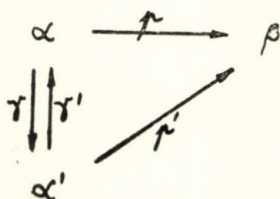
În mod analog definim $\Phi_2: \text{Hur}(E_2) \times B_2^I \longrightarrow \text{Hur}(E_2)^I$ prin $\Phi_2((e_2, \lambda_2), \mu_2)(t) = (e_2, \mu_{2t} \circ \lambda_2)$ cu $\mu_{2t}(\tau) = \mu_2(t\tau)$. Să arătăm că are loc diagrama comutativă

$$\begin{array}{ccc} \overline{\text{Hur}(E_1) \times B_1^I} & \xrightarrow{\alpha' \times \beta^I} & \overline{\text{Hur}(E_2) \times B_2^I} \\ \Phi_1 \downarrow & & \downarrow \Phi_2 \\ \text{Hur}(E_1)^I & \xrightarrow{\alpha'^I} & \text{Hur}(E_2)^I \end{array}$$

Avem $\alpha'^I \Phi_1((e_1, \lambda_1), \mu_1)(t) = \alpha' \Phi_1((e_1, \lambda_1), \mu_1)(t) = \alpha'(e_1, \mu_{1t} \circ \lambda_1) = (\alpha(e_1), \beta(\mu_{1t} \circ \lambda_1))$ și $\Phi_2(\alpha' \times \beta^I)((e_1, \lambda_1), \mu_1)(t) = \Phi_2((\alpha(e_1), \beta\lambda_1), \beta\mu_1)(t) = (\alpha(e_1), (\beta\mu_1)_t \circ \beta\lambda_1)$. Dar $\beta(\mu_{1t} \circ \lambda_1) = (\beta\mu_1)_t \circ \beta\lambda_1$, cum se verifică imediat. Deci $\alpha'^I \Phi_1 = \Phi_2(\alpha' \times \beta^I)$ și cu teorema 1 rezultă că $\text{Hur}(p)$ este o fibrare Hurewicz în C_p .

3. Fie acum $p: \alpha \longrightarrow \beta$ și $p': \alpha' \longrightarrow \beta$ două fibrări Hurewicz în C_p avînd același obiect bază β .

Definiția 4. Două fibrări p și p' avînd același obiect bază, se numesc echivalente homotopic dacă au loc diagramele comutative (în C_p)



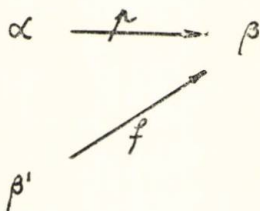
și astfel ca $\gamma' \gamma \simeq 1$, $\gamma' \gamma \simeq 1$, $p' \gamma = p$ și $p \gamma' = p$.

Echivalența homotopică a fibrărilor în categoria C_p este o relație de echivalență algebrică, cum se verifică imediat.

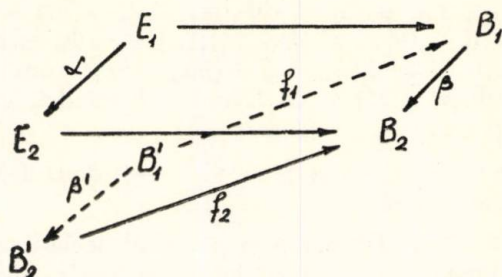
Să notăm prin $L_p(\beta)$ mulțimea claselor de fibrări în C_p echivalente homotopic, care au obiectul bază β .

Teorema 4. Asocierea $\beta \longrightarrow L_p(\beta)$ definește un functor contravariant $L_p: C_p \longrightarrow \text{Ens}$.

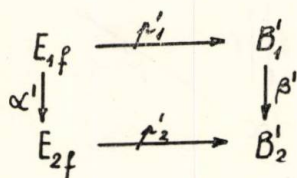
Demonstrație. Fie $f: \beta' \longrightarrow \beta$ un morfism în categoria C_p . Să definim aplicația $L_p(f): L_p(\beta) \longrightarrow L_p(\beta')$. Fie pentru aceasta $[p] \in L_p(\beta)$ și fie diagrama



adică explicitat diagramele comutative

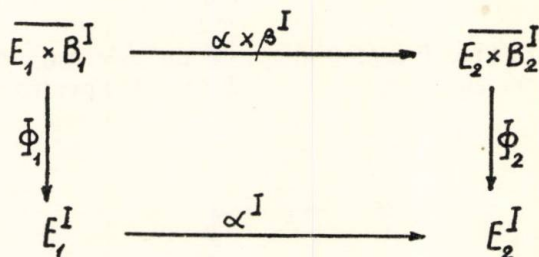


Să definim $E_{if} = (e_i, f_i)/p_i(e_i) = f_i(b_i)$ și fie diagrama comutativă



$$\alpha'(e_1, b'_1) = (\alpha(e_1), \beta'(b'_1)).$$

Să arătăm că $p' = (f'_1, p'_2): \alpha' \longrightarrow \beta'$ este o fibrare în C_p . Vom utiliza pentru aceasta teorema 1, adică vom arăta că pentru p' există un morfism de ridicare. Deoarece $p = (p_1, p_2)$ este o fibrare în C_p există pentru aceasta un morfism de ridicare $\Phi = (\Phi_1, \Phi_2)$ încît $\Phi_i(e_i, \lambda_i)(0) = e_i$, $p_i \Phi_i(e_i, \lambda_i) = \lambda_i$ și cu următoarea diagramă comutativă:



Fie $((e_1, b'_1), \lambda'_1) \in \overline{E_{1f}} \times \overline{B_1^I}$ deci $p_1(e_1) = f_1(b'_1)$ și $\lambda'_1(0) = b'_1$. Să observăm atunci că $(e_1, f_1 \lambda'_1) \in \overline{E_1} \times \overline{B_1^I}$ și deci putem defini $\Phi'_1: \overline{E_{1f}} \times \overline{B_1^I} \rightarrow \overline{E_1^I}$ prin $\Phi'_1((e_1, b'_1), \lambda'_1)(t) = \Phi_1(e_1, f_1 \lambda'_1)(t), \lambda'_1(t)$. Avem $\Phi'_1((e_1, b'_1), \lambda'_1)(0) = (\Phi_1(e_1, f_1 \lambda'_1)(0), \lambda'_1(0)) = (e_1, b'_1)$ și $p'_1 \Phi'_1((e_1, b'_1), \lambda'_1)(t) = p'_1(\Phi_1(e_1, f_1 \lambda'_1)(t), \lambda'_1(t)) = \lambda'_1(t)$.

În mod analog definim $\Phi'_2: \overline{E_{2f}} \times \overline{B_2^I} \longrightarrow E_{2f}^I$ prin $\Phi'_2((e_2, b_2), \lambda'_2)(t) = (\Phi_2(e_2, f_2 \lambda'_2)(t), \lambda'_2(t))$. Să mai arătăm că are loc diagrama comutativă:

$$\begin{array}{ccc} \overline{E_{1f}} \times \overline{B_1^I} & \xrightarrow{\alpha' \times \beta^I} & \overline{E_{2f}} \times \overline{B_2^I} \\ \Phi_1^I \downarrow & & \downarrow \Phi_2^I \\ E_{1f}^I & \xrightarrow{\alpha^I} & E_{2f}^I \end{array}$$

Avem $\alpha'^I \Phi_1^I((e_1, b_1), \lambda'_1)(t) = \alpha' \Phi_1^I((e_1, b_1), \lambda'_1)(t) = \alpha'(\Phi_1(e_1, f_1 \lambda'_1)(t), \lambda'_1(t)) = (\alpha \Phi_1(e_1, f_1 \lambda'_1)(t), \beta' \lambda'_1(t))$ și $\Phi_2^I(\alpha' \times \beta^I)((e_1, b_1), \lambda'_1)(t) = \Phi_2^I(\alpha'(e_1, b_1), \beta' \lambda'_1)(t) = \Phi_2((\alpha(e_1), \beta'(b_1)), \beta' \lambda'_1)(t) = (\Phi_2(\alpha(e_1), f_2 \beta' \lambda'_1)(t), \beta' \lambda'_1(t))$. Din egalitatea $\alpha^I \Phi_1 = \Phi_2(\alpha \times \beta^I)$ pentru $(e_1, f_1 \lambda'_1) \in \overline{E_1} \times \overline{B_1^I}$ obținem $\alpha \Phi_1(e_1, f_1 \lambda'_1)(t) = \Phi_2(\alpha(e_1), f_2 \beta' \lambda'_1)(t)$ și deci $\alpha'^I \Phi_1^I = \Phi_2^I(\alpha' \times \beta^I)$, ceea ce încheie demonstrația afirmației că p' este fibrare.

Notăm $p' = f^*p$. Se verifică cu ușurință că dacă $\bar{p} \in [p]$ atunci $f^* \bar{p} \in [f^*p]$ și deci putem defini $L_p(f)[p] = [f^*p]$. Urmînd calcule asemănătoare celor care definesc functorul $LF: \text{Top} \longrightarrow \text{Ens}$ [4] se arată că L_p este în adevăr un functor contravariant.

Se poate arăta că dacă $f \simeq f'$; $\beta' \longrightarrow \beta$ atunci $[f^*p] = [f'^*p]$ și deci poate fi considerat functorul contravariant $L'_p: C_p^h \longrightarrow \text{Ens}$ unde C_p^h este categoria perechilor cu morfisme clase de morfisme în C_p echivalente homotopic, $L'_p(\beta) = L_p(\beta)$ și $L'_p[f] = L_p(f)$.

Primită la 23 iulie 1970

Institutul politehnic, Iași,
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THE FIBRATIONS IN THE CATEGORY OF PAIRS (C_p)

(Summary)

This Note studies fiber maps in the category C_p of pairs; using the notion of lifting morphism, it gives necessary and sufficient conditions such that a morphism in category C_p of pairs be a fiber map; also, it establishes the connexion with weak fibrations [3].

The author defines a contravariant functor which associates to an object of C_p , the set of fiber homotopy equivalent classes with the given object as basis.



A NOTE ON THE LAGRANGE EXPANSION FORMULA¹⁾

BY

L. CARLITZ

1. Let

$$(1.1) \quad w = \frac{z}{\Phi(z)}, \quad (\Phi(0) \neq 0),$$

where $\Phi(z)$ is analytic about $z=0$. Then the Lagrange expansion formula [1, p. 128], [2, p. 125] reads as follows:

$$(1.2) \quad z = \sum_{n=1}^{\infty} \frac{w^n}{n!} D_0^{n-1} (\Phi(x))^n,$$

where

$$D_0^{n-1} (\Phi(x))^n = \left[\frac{d^{n-1}}{dx^{n-1}} (\Phi(x))^n \right]_{x=0}.$$

More generally, if $f(z)$ is also analytic about $z=0$, then

$$(1.3) \quad f(z) = f(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D_0^{n-1} [f'(x)(\Phi(x))^n].$$

There is also the variant

$$(1.4) \quad \frac{f(z)}{1 - w\Phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_0^n [f(x)(\Phi(x))^n].$$

It is easily verified that if we differentiate (1.3) with respect to z and then replace $f'(z)\Phi(z)$ by $f(z)$, we get (1.4). Conversely if we put

$$g(z) = \frac{f(z)}{1 - w\Phi'(z)} = \frac{f(z)\Phi(z)}{\Phi(z) - z\Phi'(z)}$$

then, since

1) Supported in part by NSF grant GP-7855.

$$\begin{aligned}
D_0^n[f(x)(\Phi(x))^n] &= D_0^n\left[\left(1 - x \frac{\Phi'(x)}{\Phi(x)}\right) g(x)(\Phi(x))^n\right] = \\
&= D_0^n[g(x)(\Phi(x))^n] - D_0^n[xg(x)\Phi'(x)(\Phi(x))^{n-1}] = \\
&= D_0^{n-1}[g'(x)(\Phi(x))^n + ng(x)\Phi'(x)(\Phi(x))^{n-1}] - nD_0^{n-1}[g(x)\Phi'(x)(\Phi(x))^{n-1}] = \\
&= D_0^{n-1}[g'(x)(\Phi(x))^n],
\end{aligned}$$

it follows that (1.4) implies

$$g(z) = g(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} D_0^{n-1}[g'(x)(\Phi(x))^n].$$

Hence (1.3) and (1.4) are equivalent.

Schur, in a posthumous paper [3], has given an arithmetic proof of these formulas.

In the present Note we shall give another simple arithmetic proof, that is, a proof that involves only manipulation of formal power series.

It evidently suffices to prove (1.4). Moreover it suffices to take

$$(1.5) \quad f(z) = z^k, \quad (k = 0, 1, 2, \dots).$$

2. There is no loss in generality in assuming that $\Phi(0) = 1$. Put

$$(2.1) \quad \Phi(x) = 1 - a_1x - a_2x^2 - \dots$$

and

$$(2.2) \quad \frac{1}{\Phi(x)} = 1 + b_1x + b_2x^2 + \dots$$

Consider $\frac{1}{\Phi(x)(1-y)} = \sum_{m,n=0}^{\infty} b_m x^m y^n$, ($b_0 = 1$). Replacing y by $x^{-1}y$ this becomes

$$(2.3) \quad \frac{1}{\Phi(x)(1-x^{-1}y)} = \sum_{m,n=0}^{\infty} b_m x^{m-n} y^n.$$

Let k denote a fixed nonnegative integer. Then that part of the right hand side of (2.3) that contains terms in x^{-k} is equal to

$$(2.4) \quad \sum_{m=0}^{\infty} b_m y^{m+k} = \frac{y^k}{\Phi(y)}.$$

On the other hand, since

$$\begin{aligned}
(1 - a_1x - a_2x^2 - \dots)(1 - x^{-1}y) &= (1 + a_1y) - x^{-1}y - \\
&- (a_1 - a_2y)x - (a_2 - a_3y)x^2 - \dots,
\end{aligned}$$

it follows that

$$\begin{aligned} \frac{1}{\Phi(x)(1-x^{-1}y)} &= \sum_{n=0}^{\infty} \frac{[x^{-1}y + (a_1 - a_2y)x + (a_2 - a_3y)x^2 + \dots]^n}{(1 + a_1y)^{n+1}} = \\ &= \sum_{r=0}^{\infty} \sum_{s_j=0}^{\infty} \frac{(r + s_1 + s_2 + \dots)!}{r! s_1! s_2! \dots} \frac{y^r (a_1 - a_2y)^{s_1} (a_2 - a_3y)^{s_2} \dots}{(1 + a_1y)^{r+s_1+s_2+\dots+1}} x^{-r+s_1+2s_2+3s_3+\dots} \end{aligned}$$

The part of the multiple sum on the right that contains terms in x^{-k} is obtained by taking

$$(2.5) \quad r = k + s_1 + 2s_2 + 3s_3 + \dots$$

Comparison with (2.4) therefore yields the following identity:

$$\begin{aligned} (2.6) \quad & \sum_{s_j=0}^{\infty} \frac{(k + 2s_1 + 3s_2 + 4s_3 + \dots)!}{s_1! s_2! \dots (k + s_1 + 2s_2 + 3s_3 + \dots)!} \times \\ & \times \frac{y^{s_1+2s_2+3s_3+\dots} (a_1 - a_2y)^{s_1} (a_2 - a_3y)^{s_2}}{(1 + a_1y)^{2s_1+3s_2+4s_3+\dots}} = \\ & = \frac{(1 + a_1y)^{k+1}}{1 - a_1y - a_2y^2 - \dots} \end{aligned}$$

If we put $z_j = a_j y^j$, ($j = 1, 2, 3, \dots$), (2.6) becomes

$$\begin{aligned} (2.7) \quad & \sum_{s_j=0}^{\infty} \frac{(k + 2s_1 + 3s_2 + 4s_3 + \dots)!}{s_1! s_2! \dots (k + s_1 + 2s_2 + 3s_3 + \dots)!} \frac{(z_1 - z_2)^{s_1} (z_2 - z_3)^{s_2} \dots}{(1 + z_1)^{2s_1+3s_2+4s_3+\dots}} = \\ & = \frac{(1 + z_1)^{k+1}}{1 - z_1 - z_2 - \dots} \end{aligned}$$

Now put $z_j - z_{j-1} = u_j$, ($j = 1, 2, 3, \dots$), so that $z_j = u_j + u_{j+1} + u_{j+2} + \dots$ ($j = 1, 2, 3, \dots$) and we get

$$\begin{aligned} (2.8) \quad & \sum_{s_j=0}^{\infty} \frac{(k + 2s_1 + 3s_2 + 4s_3 + \dots)!}{s_1! s_2! \dots (k + s_1 + 2s_2 + 3s_3 + \dots)!} \frac{u_1^{s_1} u_2^{s_2}}{(1 + u_1 + u_2 + \dots)^{2s_1+3s_2+4s_3+\dots}} = \\ & = \frac{(1 + u_1 + u_2 + \dots)^{k+1}}{1 - u_1 - 2u_2 - 3u_3 - \dots}, \end{aligned}$$

where it is assumed that $\sum u_n$ is absolutely convergent and that $|u_1| + |u_2| + |u_3| + \dots$ is sufficiently small.

In the next place, since

$$\sum_{s_0=0}^{\infty} \left(\frac{k+s_0+2s_1+3s_2+\dots}{s_0} \right) \frac{u_0^{s_0}}{(1+u_0+u_1+u_2+\dots)^{k+s_0+2s_1+3s_2+\dots+1}} =$$

$$\frac{1}{(1+u_0+u_1+u_2+\dots)^{k+2s_1+3s_2+\dots+1}} \left(1 - \frac{u_0}{1+u_0+u_1+u_2+\dots} \right)^{-k-2s_1-3s_2-\dots-1} =$$

$$= (1+u_1+u_2+\dots)^{-k-2s_1-3s_2-\dots-1},$$

it follows that

$$\sum_{s_j=0}^{\infty} \frac{(k+s_0+2s_1+3s_2+\dots)!}{s_0!s_1!s_2!\dots(k+s_1+2s_2+\dots)!} \frac{u_0^{s_0}u_1^{s_1}u_2^{s_2}\dots}{(1+u_0+u_1+u_2+\dots)^{k+s_0+2s_1+3s_2+\dots+1}} =$$

$$= \sum_{s_j=0}^{\infty} \frac{(k+2s_1+3s_2+\dots)!}{s_1!s_2!\dots(k+s_1+2s_2+\dots)!} \frac{u_1^{s_1}u_2^{s_2}}{(1+u_1+u_2+\dots)^{k+2s_1+3s_2+\dots+1}}.$$

Thus (2.8) may be replaced by

$$\sum_{s_j=0}^{\infty} \frac{(k+s_0+2s_1+3s_2+\dots)!}{s_0!s_1!s_2!\dots(k+s_1+2s_2+\dots)!} \frac{u_0^{s_0}u_1^{s_1}u_2^{s_2}}{(1+u_0+u_1+u_2+\dots)^{s_0+2s_1+3s_2+\dots+1}} =$$

$$(2.9) \quad = \frac{(1+u_0+u_1+u_2+\dots)^{k+1}}{1-u_1-2u_2-3u_3-\dots}.$$

It will be convenient to change the notation slightly. Replacing s_j by s_{j+1} and u_j by u_{j+1} , (2.9) becomes

$$\sum_{s_j=0}^{\infty} \frac{(k+s_1+2s_2+3s_3+\dots)!}{s_1!s_2!s_3!\dots(k+s_2+2s_3+\dots)!} \frac{u_1^{s_1}u_2^{s_2}u_3^{s_3}}{(1+u_1+u_2+u_3+\dots)^{s_1+2s_2+3s_3+\dots+1}} =$$

$$(2.10) \quad = \frac{(1+u_1+u_2+u_3+\dots)^{k+1}}{1-u_2-2u_3-3u_4-\dots}.$$

3. We shall now show that (2.10) implies

$$(3.1) \quad \frac{z^k}{1-w\Phi'(z)} = \sum_{n=k}^{\infty} \frac{w^n}{n!} D_0^n[x^k(\Phi(x))^n], \quad (k=0,1,2,\dots).$$

Since $D_0^n[x^k(\Phi(x))^n] = k! \binom{n}{k} D_0^{n-k}(\Phi(x))^n$, it is clear that (3.1) may be replaced by

$$(3.2) \quad \frac{(\Phi(z))^k}{1-w\Phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_0^n(\Phi(x))^{n+k}.$$

Now if $\Phi(x) = a_0 + a_1x + a_2x^2 + \dots$, ($a_0 = 1$) (for convenience we drop the negative signs in (2.1)), we have

$$(\Phi(x))^{n+k} = \sum_{s_0+s_1+s_2+\dots=n+k} \frac{(n+k)!}{s_0!s_1!s_2!\dots} a_1^{s_1} a_2^{s_2} a_3^{s_3} \dots x^{s_1+2s_2+3s_3}.$$

It follows that

$$D_0^n(\Phi(x))^{n+k} = n! \sum \frac{(n+k)!}{s_0!s_1!s_2!\dots} a_1^{s_1} a_2^{s_2} a_3^{s_3} \dots,$$

where the summation is over all s_j such that

$$s_0 + s_1 + s_2 + \dots = n + k; \quad s_1 + 2s_2 + 3s_3 + \dots = n.$$

Thus $s_0 = k + s_2 + 2s_3 + 3s_4 + \dots$ and we have

$$(3.3) \quad D_0^n(\Phi(x))^{n+k} = n! \sum \frac{(k+s_1+2s_2+3s_3+\dots)!}{s_1!s_2!\dots(k+s_2+2s_3+\dots)!} a_1^{s_1} a_2^{s_2} a_3^{s_3} \dots,$$

where the summation is over $s_1 + 2s_2 + 3s_3 + \dots = n$.

It follows that

$$(3.4) \quad \sum_{n=0}^{\infty} \frac{w^n}{n!} D_0^n(\Phi(x))^{n+k} = \\ = \sum_{s_j=0}^{\infty} \frac{(k+s_1+2s_2+3s_3+\dots)!}{s_1!s_2!\dots(k+s_2+2s_3+\dots)!} \frac{a_1^{s_1} a_2^{s_2} a_3^{s_3} \dots x^{s_1+2s_2+3s_3+\dots}}{(\Phi(z))^{s_1+2s_2+3s_3+\dots}}.$$

On the other hand, if in (2.10) we take $u_j = a_j z^j$, ($j = 1, 2, 3, \dots$), we get

$$\sum_{s_j=0}^{\infty} \frac{(k+s_1+2s_2+3s_3+\dots)!}{s_1!s_2!\dots(k+s_2+2s_3+\dots)!} \frac{(a_1 z)^{s_1} (a_2 z^2)^{s_2} (a_3 z^3)^{s_3} \dots}{(\Phi(z))^{s_1+2s_2+3s_3+\dots}} = \\ = \frac{(\Phi(z))^{k+1}}{\Phi(z) - z\Phi'(z)} = \frac{(\Phi(z))^k}{1 - w\Phi'(z)},$$

so that (3.4) becomes

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} D_0^n(\Phi(x))^{n+k} = \frac{(\Phi(z))^k}{1 - w\Phi'(z)}.$$

This evidently completes the proof.

Received July 17, 1970

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ASUPRA FORMULEI DE DEZVOLTARE ÎN SERIE A LUI LAGRANGE

(Rezumat)

Pornind de la rezultatele clasice prezentate în volumele lui A. Hurwitz și R. Courant [1] și G. Pólya și G. Szegő [2], privitoare la formula de dezvoltare în serie a lui Lagrange, se dă o demonstrație simplă a acestei formule, diferită de cea a lui I. Schur [3].

НЕКОТОРЫЕ ВОПРОСЫ ТЕОРИИ И ПРИМЕНЕНИЯ H_q -ПОЛИНОМОВ И G_q -ФУНКЦИЙ¹⁾

Б. А. БОНДАРЕНКО

Назовем H_q -полиномами многочлены, которые в трехмерном случае имеют вид

$$(1) \quad H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q}(x, y, z) = \sum_{i, j, k} (-1)^i \binom{i+j-m}{q-1} \times \\ \times \binom{i}{i+j-m} \binom{n}{2i+\alpha, 2j+\beta} x^{2i+\alpha} y^{2j+\beta} z^{2k+\delta-\alpha-\beta}, \\ i+j+k=l, \quad m=0, \quad l-q+|\alpha-\beta|(1-\delta)+\alpha\beta+1.$$

Здесь n — степень полинома, m — порядок полинома, индекс q указывает на то, что полином удовлетворяет уравнению $\Delta^q u = 0$ и неравенству $\Delta^{q-1} u \neq 0$; α и β принимают независимо друг от друга значения либо 0, либо 1, $l = \left[\frac{n}{2} \right]$, $\delta = 2 \left\{ \frac{n}{2} \right\}$, $\gamma = |\alpha - \beta| + (-1)^{\alpha+\beta} \delta$; символы $\binom{a}{b}$ обозначают, как обычно, биномиальные, а $\binom{a}{b, c}$ — полиномиальные коэффициенты.

По формуле (1) определяются в зависимости от четности n , значений α и β восемь классов многочленов всевозможной четности относительно переменных x , y и z .

H_q -полиномы линейно независимы и при $q = 1, p$ образуют в классе однородных p -гармонических многочленов степени $n \geq 2p$ базисное множество из $p(2n - 2p + 3)$ многочленов,

¹⁾ Из доклада, сделанного в Ясском университете и Ясском политехническом институте, 1 декабря 1970.

Путем преобразования инверсии относительно единичной сферы H_q -полиномы преобразуются в однородные функции степени $-n-1$ $\{G_{2-\alpha, 2-\beta, 2-\gamma}^{-n-1, m, q}(x, y, z)\}$, которые так же как и H_q -полиномы удовлетворяют уравнению $\Delta^q u = 0$ и неравенству $\Delta^{q-1} u \neq 0$. Эти функции будем называть G_q -функциями.

G_q -функции линейно независимы и при $q = \overline{1, p}$ образуют в классе однородных p -гармонических функций степени $-n-1$, получающихся в результате преобразования инверсии любого p -гармонического многочлена степени n , базисное множество из $p(2n+2p-1)$ функций. Разность числа G_q -функций степени $-n-1$ и H_q -полиномов степени n при $q = \overline{1, p}$ инвариантна относительно степени и равна $4p(p-1)$.

Если в формуле (1) положить $q = 1$, то получим класс гармонических многочленов и соответственно, класс гармонических функций. Они отличны от известных шаровых и в декартовых координатах обладают более удобными аналитическими и вычислительными свойствами. Представления гармонических многочленов в декартовых координатах, отличающиеся по форме и свойствам от шаровых многочленов и H_q -полиномов, даны М. Х. Проттером [1], П. П. Теодореску [2], А. И. Лурье [3] и др. Е. П. Майлс и Е. Вильямс [4] получили операторную форму полигармонических полиномов. Между шаровыми функциями, эллипсоидальными гармоническими функциями, многочленами др. авторов и H_q -полиномами установлены связи.

Для H_q -полиномов и G_q -функций построены рекуррентные соотношения, формулы дифференцирования и неопределенного интегрирования по каждой из переменных, формулы неопределенного интегрирования произведений любых двух H_q -полиномов или G_q -функций. Разработаны методы определенного интегрирования по многогранникам, произвольно ориентированным в пространстве, а также по границам некоторых двусвязных областей (параллелепипед со сферическим или эллипсоидальным включением и т. п.). Получены асимптотические формулы и различные соотношения для H_q -полиномов и G_q -функций. Ниже, в качестве иллюстрации, приведены некоторые из упомянутых формул для H_q -полиномов.

$$(2) \quad \Delta H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q} = -n(n-1) H_{2-\alpha, 2-\beta, 2-\gamma}^{n-2, m, q-1},$$

$$(3) \quad H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q} = \left(\frac{n}{2m+\beta} \right)^{1-\alpha} [(-1)^{1-\alpha} x H_{1+\alpha, 2-\beta, 2-\gamma}^{n-1, m+\alpha-1, q} + \\ + y H_{2-\alpha, 1+\beta, 2-\gamma}^{n-1, m+\beta-1, q}] + \alpha z H_{2-\alpha, 2-\beta, 1+\gamma}^{n-1, m, q},$$

$$(4) \quad \frac{\partial}{\partial x} H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q} = (-1)^{1-\alpha} n [H_{1+\alpha, 2-\beta, 2-\gamma}^{n-1, m, q} + \\ + (1-\alpha)(H_{1+\alpha, 2-\beta, 2-\gamma}^{n-1, m-1, q} + H_{1+\alpha, 2-\beta, 2-\gamma}^{n-1, m, q-1})],$$

$$(5) \quad \frac{\partial}{\partial y} H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q} = n H_{2-\alpha, 1+\beta, 2-\gamma}^{n-1, m+\beta-1, q},$$

$$(6) \quad \frac{\partial}{\partial z} H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q} = n H_{2-\alpha, 2-\beta, 1+\gamma}^{n-1, m, q},$$

$$(7) \quad x^{2a+\alpha} y^{2b+\beta} z^{2c+\gamma} = - \binom{n}{2a+\alpha, 2b+\beta}^{-1} \times \\ \times \sum_{q, m} (-1)^q \binom{q-1}{a} \binom{q-a-1}{b-m} H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q}, \\ a+b+c=l.$$

Формулы, установленные для Gq -функций, несколько отличаются от формул, приведенных выше, например

$$(8) \quad \Delta G_{2-\alpha, 2-\beta, 2-\gamma}^{-n-1, m, q} = -(2q-2)(2n+2q-1) G_{2-\alpha, 2-\beta, 2-\gamma}^{-n-3, m, q-1}.$$

Рассмотрены также вопросы ортогонализации Hq -полиномов по единичной сфере и кубу; доказана возможность равномерной аппроксимации p -гармонической функции удовлетворяющей некоторым условиям гладкости, посредством Hq -полиномов для случая, когда область ограничена поверхностью Ляпунова; при этом использованы результаты И. Н. Векуа [5], М. Николеску [6] и свойства Hq -полиномов.

Изучение структуры Hq -полиномов и Gq -функций дало возможность ввести т. н. *матрицы показателей* и *матрицы коэффициентов*. Это значит, что каждому Hq -полиному и любой Gq -функции поставлены в однозначное соответствие две прямоугольные числовые матрицы. Например, для полинома

$$H_{2, 2, 2}^{6, 1, 1} = 15y^4z^2 - 90x^2y^2z^2 + 15x^4z^2 - 15x^2y^4 + 30x^4y^2 - 3x^6$$

матрица показателей $A_{2, 2, 2}^{6, 1, 1}$ и матрица коэффициентов $B_{2, 2, 2}^{6, 1, 1}$ имеют вид

$$A_{2, 2, 2}^{6, 1, 1} = \begin{bmatrix} 042 & 222 & 402 \\ 240 & 420 & 600 \end{bmatrix}, \quad B_{2, 2, 2}^{6, 1, 1} = \begin{bmatrix} 15 & -90 & 15 \\ -15 & 30 & -3 \end{bmatrix}.$$

Найден простой способ построения матриц $A_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q}$ и удобный алгоритм вычисления элементов матриц $B_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, q}$. Для этих матриц, а также матриц соответствующих Gq -функциям, разработано специальное исчисление, которое позволяет арифметические, алгебраические и другие операции над Hq -полиномами и Gq -функциями свести к действиям над матрицами и эффективно использовать современные вычислительные средства.

Результаты, полученные для трехмерных Hq -полиномов и Gq -функций обобщаются на случай любого числа переменных.

Hq -полиномы и Gq -функции могут быть использованы для решения различных задач математической физики. Здесь мы остановимся на результатах применения Hq -полиномов и Gq -функций к решению трехмерных статических задач теории упругости.

Изучением полиномиальных решений уравнения Ламе

$$(9) \quad (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \Delta \mathbf{u} = 0$$

занимались С. Бергман [7], А. И. Лурье [3] и др. С. Бергман использовал для этой цели формулу Е. Альманси и шаровые функции, А. И. Лурье применил формулу М. Г. Слободянского и свои гармонические многочлены, построенные символическим методом.

Ниже дано представление составляющих вектора перемещения

$$(10) \quad \mathbf{u}_{\alpha, \beta}^{n, m; s} = \{P_{\alpha, \beta}^{n, m; s}, Q_{\alpha, \beta}^{n, m; s}, R_{\alpha, \beta}^{n, m; s}\}$$

в виде линейных комбинаций Hq -полиномов, $q = 1, 2$, полученное методом рекуррентных соотношений.

$$(11) \quad \left\{ \begin{aligned} P_{\alpha, \beta}^{n, m; s} &= \frac{1}{2} (s-2)(s-3) H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m+\alpha\beta-\beta, 1} + \frac{1}{2} (s-1)(s-2) \times \\ &\quad \times [(-1)^{\alpha} H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m, 2} - \alpha H_{2-\alpha, 2-\beta, 2-\gamma}^{n, m-1, 2}], \\ Q_{\alpha, \beta}^{n, m; s} &= \frac{1}{2} (s-2)(s-3) [(-1)^{\alpha} H_{1+\alpha, 1+\beta, 2-\gamma}^{n, m+\alpha\beta, 1} + (1-\alpha) \times \\ &\quad \times H_{1+\alpha, 1+\beta, 2-\gamma}^{n, m-1, 1}] - (s-1)(s-3) H_{1+\alpha, 1+\beta, 2-\gamma}^{n, m-\alpha\beta+\beta, 1} + \\ &\quad + \frac{1}{2} (s-1)(s-2) H_{1+\alpha, 1+\beta, 2-\gamma}^{n, m+\beta-1, 2}, \\ R_{\alpha, \beta}^{n, m; s} &= (s-1)(s-3) H_{1+\alpha, 2-\beta, 1+\gamma}^{n, m-\alpha\beta, 1} + \frac{1}{2} (s-1)(s-2) \times \\ &\quad \times \left[\frac{(1-\alpha)\lambda + (2-\alpha)\mu}{\lambda + \mu} H_{1+\alpha, 2-\beta, 1+\gamma}^{n, m, 1} + H_{1+\alpha, 2-\beta, 1+\gamma}^{n, m, 2} \right]. \end{aligned} \right.$$

Здесь $s = 1, 3$, причем порядок m изменяется в следующих пределах:

$$m = \overline{0, l + |\alpha - \beta| \delta - \alpha}, \text{ при } s = 1,$$

$$m = \overline{0, l + (1 - |\alpha - \beta|) \delta + \beta - 1}, \text{ если } s = 2,$$

$$m = \overline{0, l + |\alpha - \beta| \delta + |\alpha\beta - \beta| - 1}, \text{ когда } s = 3.$$

С учетом изменения α , β , s и порядка m формулы (11) определяют $6n + 3$ линейно независимых решений уравнения (9). В их числе $4n + 4$ чисто гармонических и $2n - 1$ бигармонических решений. Доказано, что любое полиномиальное решение уравнения (9) может быть представлено в виде линейной комбинации векторов (10), а любое регулярное в области $\mathfrak{D}$, гомеоморфной шару и ограниченной поверхностью Ляпунова, непрерывное в $\bar{\mathfrak{D}}$ решение уравнения (9), может быть равномерно аппроксимировано в $\bar{\mathfrak{D}}$ линейной комбинацией векторов (10).

На основе формул (11), закона Гука и формул (4)...(6) без особого труда строится полиномиальный тензор напряжений.

Наряду с полиномиальными решениями построены однородные решения уравнения (9) степени $-n-1$:

$$(12) \quad \mathbf{U}_{\alpha\beta}^{-n-1, m; s} = \{ \mathbf{P}_{\alpha\beta}^{-n-1, m; s}, \mathbf{Q}_{\alpha\beta}^{-n-1, m; s}, \mathbf{R}_{\alpha\beta}^{-n-1, m; s} \}.$$

Составляющие вектора перемещения (12) и соответствующего тензора напряжений выражены в виде простых линейных комбинаций $\mathbf{Gq}$ -функций, $q = 1, 2$. Заметим, что $\mathbf{Hq}$ -полиномы и $\mathbf{Gq}$ -функции с индексом $q = 1, 2, 3, \dots$ используются для точного или приближенного представления граничных функций. Это позволяет весь процесс приближенного решения краевых задач теории упругости свести к соответствующим операциям над $\mathbf{Hq}$ -полиномами и $\mathbf{Gq}$ -функциями.

Решения (10) применялись в качестве координатных функций метода наименьших квадратов и метода коллокации для построения приближенного решения основных задач статики теории упругости для прямоугольного параллелепипеда и призм, а решения (12) использовались для исследования концентрации напряжений вблизи сферического включения. Результаты, связанные с применением $\mathbf{Hq}$ -полиномов и $\mathbf{Gq}$ -функций содержатся в работах автора [8], [9].

Поступила в 10 апреля 1971 г.

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UNELE PROBLEME ALE TEORIEI ȘI APLICĂRII POLINOAMELOR H_q ȘI FUNCȚIILOR G_q

(Rezumat)

Se studiază o clasă de polinoame poliarmonice și o clasă de funcții poliarmonice în trei variabile. Se indică aplicații la probleme spațiale de teoria elasticității.

MULTI-PARAMETER EIGENVALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

BY

B. D. SLEEMAN

1. Introduction

In this Note we survey various methods and ideas involved in relation to multi-parameter eigenvalue problems in ordinary differential equations. Typical of these problems is the following; consider the finite system of ordinary linear formally self-adjoint differential equations in the k -parameters λ_s ($s = 1, \dots, k$, $k \geq 2$),

$$(1.1) \quad \frac{d^2 y_r}{dx_r^2} + \left\{ \sum_{s=1}^k p_{rs}(x_r) \lambda_s + q_r(x_r) \right\} y_r(x_r) = 0, \quad x_r \in [0, 1], \quad (r = 1, \dots, k),$$

where $p_{rs}(x_r) \in C[0, 1]$ and real valued, $q_r(x_r) \in C[0, 1]$ and real valued, and $\det \{p_{rs}(x_r)\}_{r,s=1}^k > 0 \quad \forall x = (x_1, \dots, x_k) \in I^R$ (the Cartesian product of the k -intervals $x_r \in [0, 1]$, $r = 1, \dots, k$). In order to formulate an eigenvalue problem for (1.1) we may ask that the λ_s be chosen in order that the system (1.1) has non-trivial solutions each satisfying a set of homogeneous boundary condions, say

$$(1.2) \quad U_r(y_r(0)) = 0, \quad V_r(y_r(1)) = 0, \quad (r = 1, \dots, k).$$

Probably the most common situation in which problems of the type (1.1), (1.2) arise is when we seek to solve boundary value problems associated with Laplace's equation or Helmholtz wave equation by use of the method of separation of variables. For example suppose we wish to discuss the problem of a vibrating elliptic membrane clamped along its boundary. Application of the method of separation of variables using the elliptic coordinates,

$$(1.3) \quad x = C \cosh x_1 \cos x_2, \quad y = C \sinh x_1 \sin x_2,$$

leads to the two parameter eigenvalue problem for Mathieu equations

$$(1.4) \quad \begin{cases} \frac{d^2 y_1}{dx_1^2} + (-\lambda_1 + \lambda_2 \cosh 2x_1)y_1 = 0, \\ \frac{d^2 y_2}{dx_2^2} + (\lambda_1 - \lambda_2 \cos 2x_2)y_2 = 0, \end{cases}$$

$$(1.5) \quad y_1(x_1^*) = 0, \quad y_2(x_2) = y_2(x_2 + 2\pi).$$

Here the elliptic boundary is defined by $x_1 = x_1^*$. The problem defined by (1.5) is well known and has been treated in the text by McLachlan [16].

Separation of the wave equation in other coordinates can lead to eigenvalue problems involving the ellipsoidal wave equation, Lamé's equation, spheroidal wave equation, Whittaker-Hill equation and the parabolic cylinder equation to name but a few; C. F. Meixner and Schäfer [17], Arscott [2], Sleeman [13], [24]. All the problems cited so far are essentially two parameter eigenvalue problems ($k = 2$), however Hill [13] in his investigations of the motion of the moon discussed periodic solutions of the equation

$$(1.6) \quad \frac{d^2 y}{dx^2} + T(x)y = 0,$$

in which $T(x)$ is a truncated Fourier series, the coefficients of which being identified with the parameters λ_s in (1.1) C.f. [15]. Notice that the problem (1.1), (1.2) is given in terms of k differential equations with k -parameters and defined on k -intervals. A variant of this is to consider a single differential equation with k -parameters and defined on a single interval. The eigenvalue problem is then formulated by supplying a $k + 1$ point boundary condition. For $k = 1$ we have the usual one parameter situation, $k = 2$ is a non-trivial extension and is discussed by Arscott [3].

Multi-parameter eigenvalue problems also arise in a number of other areas of mathematics and applications, [6], which do not fall immediately into the type (1.1). For example in the problem of a homogeneous vertical beam subjected to a vertical load one is led to a boundary value problem for the fourth order differential equation

$$(1.7) \quad \frac{d^4 y}{dx^4} + 2\lambda_1 \frac{d^2 y}{dx^2} - \lambda_2 y = 0.$$

Finally it is worth mentioning the excellent survey article by Atkinson [5]. In this paper Atkinson discusses a variety of aspects of multi-parameter spectral theory and goes far beyond the immediate scope of this note.

The Classical Approach

Consider the regular Sturm-Liouville problem for (1.1), namely the case when the boundary conditions (1.2) take form

$$(2.1) \quad \begin{cases} \cos \alpha_r y_r(0) - \sin \alpha_r \frac{dy_r(0)}{dx_r} = 0, & \alpha_r \in [0, \pi), \\ \cos \beta_r y_r(1) - \sin \beta_r \frac{dy_r(1)}{dx_r} = 0, & \beta_r \in (0, \pi], \end{cases}$$

$$(r = 1, \dots, k).$$

The first detailed account of this problem is given in Ince [14], where the following theorem is proved.

Theorem 1. *The eigenvalues of the system (1.1), (1.2) form a countably infinite discrete set, lying in E^k (Euclidean k -space). In particular if $(p_1, \dots, p_k)$ is a k -tuple of non-negative integers, then there is precisely one eigenvalue of this set, say $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*) \in E^k$ such that if $\{y_r(x_r, \lambda^*)\}_{r=1}^k$ is a corresponding set of simultaneous solution of (1.1) (2.1) then $y_r(x_r, \lambda^*)$ has precisely p_r zeros in $0 < x_r < 1$, $r = 1, \dots, k$. If we supplement (1.1), (2.1) with the condition*

$$(2.2) \quad \int_{I^k} \det \{p_{rs}(x_r)\}_{r,s=1}^k \prod_{r=1}^k y_r^2(x_r) dx = 1,$$

then the eigenfunctions of the system (1.1), (2.1), (2.2) form an infinite set of functions ortho-normal on I^k with respect to the weight function $\det \{p_{rs}(x_r)\}_{r,s=1}^k$.

Ince proved this result by making successive use of the usual Sturm-Oscillation theory. Quite recently the problem was taken up by Faierman [9] whose concern was to prove certain completeness and expansion theorems. Assuming additional conditions on the coefficients $p_{rs}(x_r)$, $q_r(x_r)$ in (1.1) namely

$$(2.3) \quad \begin{cases} p_{rs}(x_r) \in C_{2p+2}[0,1], \\ q_r(x_r) \in C_{2p}[0,1], \end{cases} \quad (r, s = 1, \dots, k),$$

where P is a non-negative integer satisfying $4P \leq k \leq 4P + 3$, then

Theorem 2. *Then eigenfunctions of the system (1.1), (2.1), (2.2) are complete in $L_2[I^k]$.*

If instead of the conditions (2.3) we add

$$(2.4) \quad \begin{cases} p_{rs}(x_r) \in C_p[0,1], \\ q_r(x_r) \in C_{p-2}[0,1], \end{cases} \quad (r, s = 1, \dots, k),$$

where $P = 3k$ if k is even and $P = 3k - 1$ if k is odd, then

Theorem 3. *Let $f(x) \in C_p[I^k]$ with the property that $f(x)$ and all its partial derivatives up to and including the j -th order vanish on the boundary of I^k where $j = p - 3$ if $\alpha_r \neq 0$, $\beta_r \neq \pi$, ($r = 1, \dots, k$); $j = p - 2$ otherwise. Then we can expand $f(x)$ in terms of the eigenfunctions of the system (1.1), (2.1), (2.2) for $x = (x_1, \dots, x_k) \in I^k$*

$$f(x) = \sum_{l=1}^{\infty} \left(\int_{I^k} \det \{p_{rs}(t_r)\}_{r,s=1}^k f(t) \left(\prod_{r=1}^k y_r(t_r, \lambda^l) \right) dt \prod_{r=1}^k y_r(x_r, \lambda^l) \right),$$

where the series converges absolutely and uniformly to $f(x)$ on I^k .

Faierman's approach is novel, in that instead of considering the problem (1.1), (2.1) as it stands he considers its discrete analogue and investigates the limiting process. (C.f. Atkinson [4]).

It is natural to enquire whether Fairman's investigations could be extended to the situation when the boundary conditions (1.2) instead of being of the Sturm-Liouville type are periodicity conditions, i.e.

$$(2.5) \quad y_r(0) = y_r(1), \quad y_r'(0) = y_r'(1), \quad (r = 1, \dots, k).$$

Yet another topic is the singular Sturm-Liouville problem for (1.1). Here we consider the system either on semi-infinite intervals or we consider the case in which some of the coefficients are singular at some point in a finite interval.

So far we have discussed differential equations which are linear in the parameters. Recently, the author in some work to be published, considered the following problem

$$(2.6) \quad \frac{d^2 y}{dx^2} + \{q(x; \lambda, \mu) + r(x)\}y = 0,$$

$$(2.7) \quad \begin{cases} \cos \alpha_1 y(a) - \sin \alpha_1 y'(a) = 0, \\ \cos \alpha_2 y(b) - \sin \alpha_2 y'(b) = 0, \\ \cos \alpha_3 y(c) - \sin \alpha_3 y'(c) = 0, \end{cases}$$

in which $q(x; \lambda, \mu)$, $r(x)$ are continuous functions of $x \in [a, c] \subset I$, $b \in (a, c)$. The main result here is

Theorem 4. *Let*

(i) $q(x; \lambda, \mu)$ be a continuous function of $x, \lambda, \mu \in [a, c] \times (-\infty, \infty) \times (-\infty, \infty)$ weakly increasing λ and $r(x)$ continuous in $x \in [a, c]$;

(ii) there exist for at least one value $\bar{\mu}$ of μ a $\bar{\lambda}$ such that

$$\begin{aligned} q(x; \bar{\lambda}, \bar{\mu}) &> 0 \text{ for at least one } x \in [a, b] \\ &< 0 \quad \forall x \in [b, c]; \end{aligned}$$

(iii) there exist for at least one value $\bar{\mu}$ of μ a $\bar{\lambda}$ such that

$$\begin{aligned} q(x; \bar{\lambda}, \bar{\mu}) &< 0 \text{ for at least one } x \in [b, c] \\ &> 0 \quad \forall x \in [a, b]; \end{aligned}$$

(iv) $q(x, \lambda, \mu)$ be such that for every fixed x, λ, μ with $q(x; \lambda, \mu) \neq 0$

$$\frac{q(x; k\lambda, k\mu)}{\operatorname{sgn} \{k q(x; \lambda, \mu)\}} \rightarrow \infty \text{ monotonically as } k \rightarrow \pm \infty.$$

Then there exist eigenvalues λ, μ such that the system (2.6), (2.7) has a nontrivial solution $y(x; \lambda, \mu)$ having precisely m -zeros in (a, b) and n -zeros in (b, c) .

3. Formulation in Tensor Products of Hilbert Spaces

In the previous section the approach to the multi-parameter eigenvalue problem has been based on a classical analytic foundation. However in

the one parameter case a far more elegant and powerful treatment is to consider the problems in the setting of differential operators in separable Hilbert Spaces. There are several excellent works in this area, see for example, Naimark [22], Hellwig [12], Akhiezer and Glazman [1]. It seems natural therefore to formulate the multi-parameter eigenvalue problem in an analogous way. It turns out that here it is necessary to consider the problem in a Hilbert Space which is the tensor product of a collection of separable Hilbert Spaces. In this section we outline the formulation, a full discussion of the problem in this context will be given elsewhere.

Let H_r , ($r=1, \dots, k$) be separable Hilbert Spaces and let $u_r, v_r \in H_r$. To be specific let

$$(3.1a) \quad H_r = \left\{ u_r \left| \int_{a_r}^{b_r} |u_r(x_r)|^2 \rho_r(x_r) dx_r < \infty \right. \right\},$$

with scalar product

$$(3.1b) \quad (u_r, v_r)_r = \int_{a_r}^{b_r} u_r(x_r) \bar{v}_r(x_r) \rho_r(x_r) dx_r,$$

where in general u_r, v_r are complex valued and the real valued and continuous function

$$\rho_r(x_r) > 0 \quad \forall x_r \in [a_r, b_r].$$

(N.B. All integrals are to be interpreted in the Lebesgue sense). In H_r , ($r=1, \dots, k$) we consider the differential operators A_r in $D(A_r)$ defined by

$$(3.2) \quad A_r u_r = \frac{1}{\rho_r} \{ -(p_r u_r)' + q_r(x_r) u_r \}, \quad \forall u_r \in D(A_r),$$

with p_r, p_r', q_r real valued and continuous $\forall x_r \in [a_r, b_r]$, and

$$(3.3) \quad D(A_r) = \{ u_r | u_r \in C^2[a_r, b_r], U_r(u_r(a_r)) = V_r(u_r(b_r)) = 0 \}.$$

For the multi-parameter eigenvalue problem we are asked to consider k -eigenvalue problems in k separable Hilbert Spaces where solutions are sought in the k -domains $D(A_r)$, ($r=1, \dots, k$). That is we consider the k eigenvalue problems

$$(3.4) \quad A_r y_r = \sum_{s=1}^k \lambda_s p_{rs}(x_r) y_r, \quad y_r \in D(A_r), \quad (r=1, \dots, k),$$

where $P = \det \{ p_{rs}(x_r) \}_{r,s=1}^k > 0 \quad \forall x \in I^k$ (the Cartesian product of the intervals $[a_r, b_r]$, ($r=1, \dots, k$)).

We now form the tensor product of each Hilbert Space H_r in the manner of Murray and Von Neumann [21] and denote this tensor product by

$$(3.5) \quad H = \prod_{s=1}^k \otimes H_s.$$

In H we define, for $u, v \in H$,

$$(3.6) \quad H = \left\{ u \left| \int_{I^k} |u|^2 P \rho_1 \dots \rho_k dx < \infty \right. \right\},$$

with scalar product

$$(3.7) \quad (u, v) = \int_{I^k} v \bar{u} P \rho_1 \dots \rho_k dx.$$

The actual construction of H needs careful consideration and we refer to Murray and Von Neumann for details. For example, because the $H_r (r=1, \dots, k)$ are separable Hilbert Spaces, it is not a trivial matter to prove that H is also a separable Hilbert Space. Also if $f_r(x_r) (r=1, \dots, k)$ such that $f_r: H_r \rightarrow H_r$ is a set of endomorphisms of the H_r , then the f_r are not operators on the tensor product. However they will induce endomorphisms of H which may be denoted by

$$(3.8) \quad \tilde{f}_r = I_1 \otimes I_2 \otimes \dots \otimes I_{r-1} \otimes f_r \otimes \dots \otimes I_k,$$

where $I_r (r=1, \dots, k)$ are identity operators in H_r . Thus in (3.6), (3.7) we should strictly write $\prod_{r=1}^k \otimes \rho_r$ instead of the algebraic product given.

The problem now is to determine the analogous eigenvalue problem in H . The process for doing this is essentially to use the method of separation of variables in reverse. Formally write the system (3.4) as the matrix equation

$$(3.9) \quad \begin{pmatrix} \frac{1}{y_1} A_1 y_1 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{y_k} A_k y_k \\ y_k \end{pmatrix} = \begin{pmatrix} p_{11} & \dots & p_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_{k1} & \dots & p_{kk} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \lambda_k \end{pmatrix}.$$

Since $P = \det \{p_{rs}\}_{r,s=1}^k > 0$ and denoting by p_{ij}^* the co-factor of p_{ij} in the $k \times k$ non-singular matrix in (3.9) we rewrite (3.9) as

$$(3.10) \quad \begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \frac{p_{11}^*}{P} & \dots & \frac{p_{k1}^*}{P} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{p_{1k}^*}{P} & \dots & \frac{p_{kk}^*}{P} \end{pmatrix} \begin{pmatrix} \frac{1}{y} A_1 y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{y_k} A_k y_k \end{pmatrix}.$$

Thus we are led to consider the following system of partial differential expressions

$$(3.11) \quad \lambda_r = \frac{1}{P} \sum_{s=1}^k p_{sr}^* \frac{1}{y_s} A_s y_s, \quad (r = 1, \dots, k).$$

Formal multiplication of (3.11) by $y = y_1 \dots y_k$ leads to the partial differential equations

$$(3.12) \quad \frac{1}{P} \sum_{s=1}^k p_{sr}^* A_s y = \lambda_r y, \quad (r = 1, \dots, k).$$

This formal derivation suggests that we investigate the following partial differential operators a_r in $D(a_r) \subset H$ defined by

$$(3.13) \quad a_r Y = \frac{1}{P} \sum_{s=1}^k p_{sr}^* \tilde{A}_s Y, \quad \forall Y \in D(a_r), \quad (r = 1, \dots, k),$$

where $\tilde{A}_s = I_1 \otimes \dots \otimes I_{s-1} \otimes A_s \otimes I_{s+1} \dots \otimes I_k$, and $D(a_r) = \sum_{p=1}^k \otimes D(A_p)$ and so is the same for every member of (3.13).

By this formulation it is now possible to discuss all problems mentioned in Section 2 in a unified and much more powerful way. It is important to mention however that the eigenfunctions satisfying the eigenvalue problem formulated from (3.13) are not necessarily decomposable into tensor product form. Such decomposable tensors must be sought in a dense subspace of $D(a_r)$.

For the use of tensor products of Hilbert Spaces in a abstract formulation of separation of variables, see B. Friedman [11] and Cordes [7].

4. Numerical Treatment

Although the ideas relating to the multi-parameter eigenvalue problem as outlined in the previous sections form powerful tools for investigation, they do not in themselves suggest methods of computing eigenfunctions

or the corresponding eigenvalues. Several techniques however are available, but these are mainly restricted to certain types of eigenvalue problems and are usually dictated by special properties of the associated differential equations. Thus for example if we are treating periodic eigenvalue problems for say Mathieu's equation or Lamé's equation then in order to determine eigenvalues one resorts to computing finite or infinite continued fractions, solving numerically three term recurrence relations or finding the latent roots of tri-diagonal matrices). See [2], [17], [18], [19]. In this section we propose a general method. The discussion to follow will be restricted to the two-parameter case, ($k=2$) but there appears to be no difficulty in extending the ideas to the case $k>2$. To be specific let us consider the Sturm-Liouville eigenvalue problem

$$(4.1) \quad \begin{cases} \frac{d^2 y_1}{dx_1^2} + (\lambda_1 p_{11}(x_1) + \lambda_2 p_{12}(x_1)) y_1 = 0, \\ y_1(0) = y_1(1) = 0, \end{cases}$$

$$(4.2) \quad \begin{cases} \frac{d^2 y_2}{dx_2^2} + (\lambda_1 p_{21}(x_2) + \lambda_2 p_{22}(x_2)) y_2 = 0, \\ y_2(0) = y_2(1) = 0, \end{cases}$$

where $p_{rs} \in C[0,1]$ and real valued, $r, s = 1, 2$ and

$$\det \{p_{rs}(x_r)\}_{r,s=1}^2 > 0, \quad \forall x = (x_1, x_2) \in I^2.$$

In this statement of the problem, and bearing in mind the discussion of the previous section, we have the Hilbert Spaces H_r , ($r=1, 2$) defined by (3.1) and the domains $D(A_r)$, $r=1, 2$ as described by (3.3). By applying the tensor product construction (3.5) it is easy to see that we are led to consider the following eigenvalue problems for partial differential equations. That is we have

$$(4.3) \quad a_r Y_r = \lambda_r Y_r, \quad Y_r \in D(a_r) = D(A_1) \otimes D(A_2),$$

i.e. $Y_r = 0$ on the boundary of the unit square $[0,1] \times [0,1]$.

In most application the a_r are elliptic operators and so (4.3) defines an eigenvalue problem for elliptic partial differential operators. In the literature there exists a number of numerical methods for solving such problems and among the most recent we cite the work of Fox, Henrici, Moler, Payne and Donnelly [10], [20], [8]. However these papers are mainly concerned with eigenvalue problems for Laplace's equation under Dirichlet boundary conditions. Thus in order to make the idea outlined above a useful approach, we need to extend the analysis of the above writers to more general elliptic operators and more general (e.g. Sturm-Liouville type) boundary conditions. Having obtained numerical values of the eigenvalues

for (4.3) we stress that the associated eigenfunctions are not necessarily decomposable into product form and so in general one would have to compute the eigenfunctions directly from (4.1a), (4.2a). It is hoped to take up this aspect in subsequent papers.

Received September 12, 1970

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PROBLEME DE VALORI PROPRII MULTI-PARAMETRICE
PENTRU ECUAȚII DIFERENȚIALE ORDINARE

(Rezumat)

Se trec în revistă diferite metode și idei, printre care unele inedite ale autorului, privind problemele de valori proprii multi-parametrice pentru ecuații diferențiale ordinare. După abordarea clasică, se formulează problema într-un spațiu Hilbert, produs tensorial al unui număr de spații Hilbert separabile, și se propune un procedeu general de calcul numeric.



EIGENVALUE PROBLEMS FOR MANGERON'S POLYVIBRATING EQUATIONS (I) ¹⁾

BY

MEHMET NAMIK OĞUZTÖRELI

*This paper is dedicated to the sixty-fifth
anniversary of Professor D. MANGERON*

1. Introduction

Since D. Mangeron's habilitation thesis [1], a large number of contributions have been devoted to polyvibrating equations of Mangeron. A rather complete list of references can be found in [2]; see also [3].

In this paper we consider the Mangeron equation

$$(1.1) \quad \partial(p(x)q(y)\partial u) - \lambda r(x)s(y)u = 0,$$

for $a < x < b$, $c < y < d$, subject to the boundary conditions

$$(1.2) \quad u(a, y) = u(b, y) = 0, \quad (c \leq y \leq d),$$

and

$$(1.3) \quad u(x, c) = u(x, d) = 0, \quad (a \leq x \leq b),$$

where λ is a parameter and ∂ denotes the *total derivative operator* of M. Picone [4]:

$$(1.4) \quad \partial = \frac{\partial^2}{\partial x \partial y}.$$

We assume that $p(x)$ and $q(y)$ are continuously differentiable and positive functions in $a \leq x \leq b$ and $c \leq y \leq d$ and $r(x)$ and $s(y)$ are continuous and positive functions in $a \leq x \leq b$ and $c \leq y \leq d$, respectively.

¹⁾ This work was partly supported by the National Research Council of Canada under the Grant NRC-A-4345 through the University of Alberta.

We wish to determine the values of the parameter λ in such a manner that the boundary value problem (1.1)...(1.3) will admit a nontrivial solution in the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$. To do so we shall employ the techniques of the theory of finite integral transforms, although the same results can be established by the direct use of the method of separation of variables.

Let us note that, in the majority of practical problems, the condition $sp(x) > 0, q(y) > 0, r(x) > 0$ and $s(y) > 0$ in their domains of definition can be weakened by the requirements $p(x) \geq 0, q(y) \geq 0, r(x) \geq 0$ and $s(y) \geq 0$, respectively. Further, the conditions (1.2) clearly imply the following equalities:

$$(1.5) \quad \left. \frac{\partial}{\partial y} \left(q(y) \frac{\partial u}{\partial y} \right) \right|_{y=c} = \left. \frac{\partial}{\partial y} \left(q(y) \frac{\partial u}{\partial y} \right) \right|_{y=d} = 0, \quad (c \leq y \leq d).$$

2. Reduction to Ordinary Sturm-Liouville Problems

Let $K(x, \mu)$ be a function which is twice differentiable on $a \leq x \leq b$ depending on a parameter μ with a discrete set of values. For any integrable function $w(x)$ on $a \leq x \leq b$ we put

$$(2.1) \quad \tilde{w}(\mu) = \int_a^b K(x, \mu) r(x) w(x) dx.$$

Clearly, $\tilde{w}(\mu)$ is the finite integral transform of $w(x)$ with kernel $K(x, \mu)$ and weight $r(x)$.

Let $u = u(x, y)$ be a function continuous in R with continuous partial derivatives $\partial u, \partial \frac{\partial u}{\partial x}, \partial \frac{\partial u}{\partial y}$ and $\partial^2 u$, satisfying the boundary conditions (1.2) ... (1.3). Then, by integration by parts twice and using the equalities (1.4), we obtain

$$(2.2) \quad \int_a^b K(x, \mu) \partial(p(x)q(y)) \partial u dx = \left[p(x) K(x, \mu) \partial \left(q(y) \frac{\partial u}{\partial y} \right) \right]_{x=a}^{x=b} + \\ + \frac{\partial}{\partial y} \left\{ q(y) \frac{\partial}{\partial y} \int_a^b u(x, y) \frac{\partial}{\partial x} \left(p(x) \frac{\partial K(x, \mu)}{\partial x} \right) dx \right\}.$$

We now consider the Sturm-Liouville problem

$$(2.3) \quad \frac{d}{dx} \left(p(x) \frac{dU}{dx} \right) + \mu r(x) U = 0, \quad U(a) = U(b) = 0.$$

Let $\mu_1, \mu_2, \dots, \mu_m, \dots$ denote the eigenvalues of the Sturm-Liouville problem (2.3), and let $U_1(x), U_2(x), \dots, U_m(x), \dots$ denote the normalized (with weight $r(x)$) eigenfunctions corresponding to them. Since the functions $p(x)$ and $r(x)$ are assumed to be continuous and positive, and $p(x)$ is continuously differentiable for $a \leq x \leq b$, the functions $U_1(x), U_2(x), \dots, U_m(x), \dots$ are orthogonal with weight $r(x)$ and form a complete set of orthonormal functions on $a \leq x \leq b$:

$$(2.4) \quad \int_a^b U_j(x) U_k(x) r(x) dx = \delta_{jk}, \quad (j, k = 1, 2, 3, \dots),$$

where δ_{jk} is the Kronecker delta.

We now take $\mu_1, \mu_2, \dots, \mu_m, \dots$ for the discrete set of values of the parameter μ in the kernel of the finite integral transform (2.1) and we take $K(x, \mu_m) \equiv U_m(x)$, ($m = 1, 2, 3, \dots$). We then find

$$(2.5) \quad \int_a^b K(x, \mu_m) \partial(p(x)q(y)\partial u) dx = -\mu_m \frac{\partial}{\partial y} \left(q(y) \frac{\partial \tilde{u}(\mu_m, y)}{\partial y} \right),$$

where

$$(2.6) \quad \tilde{u}(\mu_m, y) = \int_a^b K(x, \mu_m) r(x) u(x, y) dx.$$

Thus, multiplying (1.1) and (1.3) by $K(x, \mu_m) \equiv U_m(x)$, and integrating over the interval $a \leq x \leq b$, we obtain

$$(2.7) \quad \mu_m \frac{\partial}{\partial y} \left(q(y) \frac{\partial \tilde{u}(\mu_m, y)}{\partial y} \right) + \lambda s(y) u(\mu_m, y) = 0,$$

with

$$(2.8) \quad \tilde{u}(\mu_m, c) = \tilde{u}(\mu_m, d) = 0.$$

We now consider the Sturm-Liouville problem

$$(2.9) \quad \frac{d}{dy} \left(q(y) \frac{dV}{dy} \right) + \nu s(y) V = 0, \quad V(c) = V(d) = 0, \quad \left(\nu = \frac{\lambda}{\mu_m} \right).$$

Let $\nu_1, \nu_2, \dots, \nu_n, \dots$ denote the eigenvalues of the Sturm-Liouville problem (2.9) and let $V_1(y), V_2(y), \dots, V_n(y), \dots$ denote the normalized (with weight $s(y)$) eigenfunctions corresponding to them, which form a complete set of orthonormal functions on $c \leq y \leq b$ by assumptions made about the functions $q(y)$ and $s(y)$:

$$(2.10) \quad \int_c^d V_j(y) V_k(y) s(y) dy = \delta_{jk}.$$

Thus, the Sturm-Liouville problem (2.7) ... (2.8) admits the values $\lambda_{m,n} = \mu_m \nu_n$ as eigenvalues with the corresponding eigenfunctions $V_n(y)$. It is easy to verify that the functions

$$(2.11) \quad u_{m,n}(x,y) = U_m(x) V_n(y), \quad (m,n = 1, 2, 3, \dots),$$

are eigenfunctions of the problem (1.1) ... (1.3) with eigenvalues

$$(2.12) \quad \lambda = \lambda_{m,n} = \mu_m \nu_n.$$

Clearly the functions $u_{m,n}(x,y)$ constitute a complete set of functions orthonormal with weight $r(x)s(y)$ over the rectangle R .

3. Examples

We now illustrate the above results by the following simple examples:

1° The functions

$$(3.1) \quad u_{m,n}(x,y) = \frac{2}{\pi} \sin mx \sin ny, \quad (m,n = 1, 2, 3, \dots)$$

are the eigenfunctions with the eigenvalues $\lambda = m^2 n^2$ of the boundary value problem

$$(3.2) \quad \partial^2 u - \lambda u = 0, \quad u(0,y) = u(\pi,y) = u(x,0) = u(x,\pi) = 0.$$

Clearly these functions form a complete set of orthonormal functions over the sequence $0 \leq x \leq \pi, 0 \leq y \leq \pi$.

2° The functions

$$(3.3) \quad u_{m,n}(x,y) = \frac{J_0(x\mu_m) J_0(y\nu_n)}{\left[\int_0^1 J_0^2(x\mu_m) x dx \int_0^1 J_0^2(y\nu_n) x dx \right]^{1/2}}, \quad (m,n = 1, 2, 3, \dots),$$

where $J_0(t)$ is the Bessel function of the first kind of order zero and μ_k is the k -th zero of $J_0(t)$, are the eigenfunctions with the eigenvalues $\lambda = \mu_m^2 \nu_n^2$ of the boundary value problem

$$(3.4) \quad \partial(xy\partial u) - \lambda xy u = 0, \quad u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0.$$

Obviously the functions (3.3) constitute a complete set of orthonormal functions with weight xy over the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

3° The functions

$$(3.5) \quad u_{m,n}(x,y) = \frac{\sqrt{(2m+1)(2n+1)}}{2} P_m(x)P_n(y), \quad (m,n = 0, 1, 2, \dots),$$

where $P_k(x)$ denotes the Legendre polynomial of degree k , are the eigenfunctions with eigenvalues $\lambda = mn(m+1)(n+1)$ of the boundary value problem

$$(3.6) \quad \partial[(1-x^2)(1-y^2)\partial u] - \lambda u = 0, \quad u(-1,y)u(1,y) = u(x,-1) = u(x,1) = 0,$$

and they form a complete set of orthonormal functions over the square $-1 \leq x \leq 1, -1 \leq y \leq 1$.

4. Extension

Consider the Mangeron equation

$$(4.1) \quad \partial(p\partial u) - \lambda ru = 0,$$

for $a_i < x_i < b_i$, ($i = 1, \dots, n$), subject to the conditions

$$(4.2) \quad u|_{x_i=a_i} = u|_{x_i=b_i} = 0, \quad (i = 1, \dots, n),$$

where $u = u(x_1, \dots, x_n)$, λ is a parameter, and

$$(4.3) \quad p = p(x_1, \dots, x_n) = \prod_{i=1}^n p_i(x_i), \quad r = r(x_1, \dots, x_n) = \prod_{i=1}^n r_i(x_i)$$

and

$$(4.4) \quad \partial = \frac{\partial^n}{\partial x_1 \dots \partial x_n}.$$

We assume that the functions $p_i = p_i(x_i)$ are continuously differentiable and positive, $r_i(x_i)$ are continuous and positive in $a_i \leq x_i \leq b_i$, ($i = 1, \dots, n$), respectively. Under these conditions, let $\mu_{i,k}$ ($k = 1, 2, 3, \dots$) be the eigenvalues and $U_{i,k}(x_i)$ the corresponding (normalized with weight $r_i(x_i)$) eigenfunctions of the Sturm-Liouville problem

$$(4.5) \quad \frac{d}{dx_i} \left(p_i(x_i) \frac{dU_i}{dx_i} \right) + \mu_{i,k} r_i(x_i) U_i = 0, \quad U_i(a_i) = U_i(b_i) = 0$$

for each fixed ($i = 1, \dots, n$). Clearly the functions

$$(4.6) \quad u_{k_1, k_2, \dots, k_n}(x_1, x_2, \dots, x_n) = \prod_{i=1}^n U_{i, k_i}(x_i), \quad (k_i = 1, 2, 3, \dots)$$

constitute a complete system of orthonormal system with weight

$r(x_1, \dots, x_n)$ on the domain $a_i \leq x_i \leq b_i$, ($i = 1, \dots, n$), and each of the functions $u_{k_1, \dots, k_n}$ is an eigenfunction of the boundary value problem (4.1), (4.2) with eigenvalue

$$(4.7) \quad \lambda = \lambda_{k_1, \dots, k_n} = \prod_{i=1}^n \mu_{i, k_i}.$$

Received April 10, 1971

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PROBLEME DE AUTOVALORI PENTRU ECUAȚIILE POLIVIBRANTE ALE LUI MANGERON (I)

(Rezumat)

Se utilizează tehnica transformatelor integrale finite pentru a studia o problemă de autovalori pentru ecuațiile polivibrante ale lui Mangeron.

A COMPARISON THEOREM FOR EIGENVALUES
OF MANGERON EQUATIONS¹⁾

BY

SUNDARAM EASWARAN

Dedicated to Professor D. MANGERON on his 65th birthday

1. Comparison theorems for ordinary differential equations have been extensively studied and recently several contributions have been devoted to the extension of these results to partial differential equations. A detailed list of publications in this field can be found in the book of C. A. Swanson [10] and in the recent paper of Allegretto and Swanson [1]. In this Note we give an extension of a theorem of Z. Nehari to the partial differential equations of D. Mangeron [4]

$$(1) \quad \frac{\partial^4 u}{\partial x^2 \partial y^2} - \lambda p(x, y)u = 0,$$

subject to the boundary conditions

$$(2) \quad u_{xy}(a, y) = 0, \quad u_{xy}(x, c) = 0, \quad u(b, y) = 0, \quad u(x, d) = 0,$$

where $u(x, y)$ is assumed to be a continuous function defined on $R: \{a \leq x \leq b; c \leq y \leq d\}$ with $u_x(x, y)$, $u_y(x, y)$, $u_{xy}(x, y)$, $u_{xyx}(x, y)$, $u_{xyy}(x, y)$ and $u_{xyyx}(x, y)$ continuous in R and $p(x, y)$ is a positive continuous function in the same domain.

It can easily be shown that the Green's function of the operator $\frac{\partial^4}{\partial x^2 \partial y^2}$ under the boundary conditions (2) is given by

1) This work was partly supported by the National Research Council of Canada by Grant NRC-A-4345 through the University of Alberta.

$$G(x, y; \xi, \eta) \begin{cases} (y-d)(x-b) & \xi \leq x, \eta \leq y, & (b-\xi)(d-y) & \xi \geq x, \eta \leq y, \\ (d-\eta)(b-x) & \xi \leq x, \eta \geq y, & (b-\xi)(d-\eta) & \xi \geq x, \eta \geq y; \end{cases}$$

observe that $G(x, y; \xi, \eta)$ is positive in R except on the lines $x = b$, $y = d$ as a function of x, y for fixed (ξ, η) . Thus the solution (1) and (2) is equivalent to the integral equation

$$(3) \quad u(x, y) = \lambda \iint_{a \leq \xi \leq b, c \leq \eta \leq d} G(x, y; \xi, \eta) p(\xi, \eta) u(\xi, \eta) d\xi d\eta.$$

By Jentzsch Theorem [2] the eigenfunction of this integral equation corresponding to the minimum eigenvalue is positive. Now we state and prove the following theorem which is a direct extension of Nehari's Theorem [7]:

Theorem. Let $p(x, y)$ be a positive continuous function $q(x, y)$ be a continuous function defined on R such that

$$(4) \quad \iint_{a \leq \xi \leq b, c \leq \eta \leq d} p(\xi, \eta) d\xi d\eta \leq \iint_{a \leq \xi \leq b, c \leq \eta \leq d} q(\xi, \eta) d\xi d\eta.$$

Let λ and μ respectively be the least eigenvalues of the equations

$$(A) \quad \frac{\partial^4 u}{\partial x^2 \partial y^2} - \lambda p(x, y) u = 0,$$

subject to the boundary conditions

$$(B) \quad u_{xy}(a, y) = 0, \quad u_{xy}(x, c) = 0, \quad u(b, y) = 0, \quad u(x, d) = 0$$

and the equations

$$(A') \quad \frac{\partial^4 v}{\partial x^2 \partial y^2} - \mu q(x, y) v = 0,$$

subject to the boundary conditions

$$(B') \quad v_{xy}(a, y) = 0, \quad v_{xy}(x, c) = 0, \quad v(b, y) = 0, \quad v(x, d) = 0.$$

Then we have $\mu \leq \lambda$, the equality sign holds when $p = q$.

Before we give the proof of this theorem we prove the following

Lemma. If $\varphi(x, y)$ and $\psi(x, y)$ are continuous together with $\varphi_x(x, y)$, $\varphi_y(x, y)$, $\varphi_{xy}(x, y)$ and $\psi_x(x, y)$, $\psi_y(x, y)$, $\psi_{xy}(x, y)$ on R and satisfy the condition

$$(5) \quad \varphi(a, y) = \varphi(x, c) = 0, \quad \psi(x, d) = \psi(b, y) = 0.$$

Then

$$\iint_{a,c}^{b,d} \varphi \psi_{xy} dx dy = \iint_{a,c}^{b,d} \varphi_{xy} \psi dx dy.$$

Proof. Integrating the identity $\varphi \psi_{xy} - \varphi_{xy} \psi = (\varphi \psi_x)_y - (\varphi_y \psi)_x$ and making use of the conditions (5) we find

$$\begin{aligned} \iint_{a,c}^{b,d} [\varphi \psi_{xy} - \varphi_{xy} \psi] dx dy &= \int_a^b [\varphi \psi_x(x, d) - \varphi \psi_x(x, c)] dx + \\ &+ \int_c^d [\varphi_y \psi(b, y) - \varphi_y \psi(a, y)] dy = 0, \end{aligned}$$

from which the result follows.

Proof of the Theorem. Consider the following identity due to Manaresi [3]

$$(6) \quad [uu_{xy}]_{xy} - [u_x u_{xy}]_y - [u_y u_{xy}]_x + u_{xy}^2 = uu_{xyxy}.$$

Then under the boundary conditions (B) we have

$$\iint_{a,c}^{b,d} uu_{xyxy} dx dy = \iint_{a,c}^{b,d} u_{xy}^2 dx dy.$$

Hence, if $u(x, y)$ is a solution of the problem (A) and (B), then

$$\iint_{a,c}^{b,d} u_{xy}^2 dx dy = \lambda \iint_{a,c}^{b,d} p u^2 dx dy.$$

Now we use the lemma with $\psi(x, y) = u^2(x, y)$, $\varphi(x, y) = \iint_{a,c}^{x,y} p(\xi, \eta) d\xi d\eta$

which yields

$$\iint_{a,c}^{b,d} u_{xy}^2 dx dy = \lambda \iint_{a,c}^{b,d} p u^2 dx dy = \lambda \iint_{a,c}^{b,d} \left(\iint_{a,c}^{x,y} p d\xi d\eta \right) (u^2)_{xy} dx dy.$$

Since $u(x, y)$ is the eigenfunction corresponding to the least eigenvalue we have $u(x, y) \geq 0$ by Jentzsch Theorem. Further by virtue of the bound-

ary conditions (2) we have $u_{xy}(x, y) = \lambda \iint_{a,c}^{x,y} p u dx dy \geq 0$. Thus we have

$$u_x(x, d) - u_x(x, y) = \int_y^d u_{x\eta}(x, \eta) d\eta \geq 0.$$

Further we have $u_x(x, d) = 0$ by (B), therefore $u_x(x, y) \leq 0$. Similarly we have $u_y(x, y) \leq 0$. Thus we have $(u^2)_{xy} = 2u u_{xy} + 2u_x u_y \geq 0$.

Hence

$$\begin{aligned} \iint_{a,c}^{b,d} u_{xy}^2 dx dy &= \lambda \iint_{a,c}^{b,d} \left(\iint_{a,c}^{x,d} p d\xi d\eta \right) (u^2)_{xy} dx dy \leq \\ &\leq \lambda \iint_{a,c}^{b,d} \left(\iint_{a,c}^{x,y} q d\xi d\eta \right) (u^2)_{xy} dx dy = \lambda \iint_{a,c}^{b,d} q u^2 dx dy. \end{aligned}$$

On the other hand by the virtue of the minimum property of eigenvalues, which can be proved easily

$$\mu \iint_{a,c}^{b,d} q u^2 dx dy \leq \iint_{a,c}^{b,d} u_{xy}^2 dx dy \leq \lambda \iint_{a,c}^{b,d} q u^2 dx dy,$$

which implies inequality $\mu \leq \lambda$. This completes the proof of the theorem.

Acknowledgement. The author is indebted to Professor D. Mangeron and M. N. Oguztöreli for their kind help and encouragement.

Received April 10, 1971

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TEOREMĂ DE COMPARAȚIE PENTRU AUTOVALORILE ECUAȚIILOR LUI MANGERON

(Rezumat)

Se extinde la ecuațiile lui Mangeron teorema lui Nehari privitoare la comparația autovalorilor pentru ecuații diferențiale ordinare.

OBSERVATIONS SUR QUELQUES ÉQUATIONS FONCTIONNELLES CONCERNANT DES LOIS NORMALES DE PROBABILITÉ

PAR

VIOREL MURGESCU

1. Introduction

Dans [1] on a établi une généralisation de la loi normale de Gauss-Laplace dont la densité de probabilité d'une variable aléatoire X a la forme

$$\rho_p(x) = \left[2h \Gamma \left(1 + \frac{1}{2p} \right) \right]^{-1} \exp \left[\left(-\frac{x}{h} \right)^{2p} \right], \quad (p = 1, 2, \dots).$$

Mais, il est connu de [2], [3] et [4], que pour $p = 1$ la fonction caractéristique correspondante est caractérisée par l'équation fonctionnelle $\varphi(x) = \varphi(ax) \varphi(bx)$, ($a, b > 0$; $a^2 + b^2 = 1$). On sait encore qu'une généralisation des résultats contenus dans [4] a été donnée en 1964 par Laha R. G., Lukacs E. et Rényi A. [5].

En considérant que l'extension des résultats de [4] et [5] sur les fonctions complexes de variable réelle, de la forme $\varphi(x) = \exp(\mu x^{2p})$, ($\mu = \mu_1 + i\mu_2$; $p = 1, 2, 3, \dots$), peut jouer un rôle important dans la caractérisation de la loi normale généralisée et de même, dans le problème de factorisation des fonctions caractéristiques correspondantes, dans ce qui suit on se propose de la présenter.

2. Une généralisation immédiate d'un théorème de Vincze

Théorème 1. *Soit l'équation fonctionnelle*

$$(2.1) \quad \varphi(x) = \varphi(a_1 x) \varphi(a_2 x), \quad (a_1, a_2 > 0, \quad a_1^{2p} + a_2^{2p} = 1),$$

où φ est une fonction (non constante) à valeurs complexes, de la variable réelle x , deux fois dérivable dans $x = 0$ et $p = 1, 2, 3, \dots$.

Alors,

$$(2.2) \quad \varphi(x) = \exp [\mu x^{2p}]$$

est la seule solution de l'équation (2.1), μ étant une constante complexe différente de zéro.

Vu que la démonstration de ce théorème suit les mêmes étapes que le théorème établi en [4] pour $p = 1$, on considère qu'on peut renoncer à la présenter.

3. Une généralisation du théorème 1

En poursuivant un théorème de [5], nous avons signalé dans [6] la possibilité de le généraliser aux fonctions de la forme (2.2), ce qui conduit au résultat suivant :

Théorème 2. Soit $\varphi(x)$ une fonction complexe d'une variable réelle x et soit encore $\{a_k\}$ une suite réelle de nombres non négatifs, satisfaisant la condition

$$(3.1) \quad \sum_{k=1}^{\infty} a_k^{2p} = 1, \quad (0 < a_k < 1, \quad k \in \mathbb{N}).$$

Supposons que pour $p, r \in \mathbb{N}$ donnés, ($p \leq r \leq 2p-1$), il existe les constantes complexes λ et μ de sorte que

$$(3.2) \quad \lim_{x \rightarrow 0} \frac{\varphi(x) - \varphi(0) - \lambda x^r}{x^{2p}} = \mu,$$

et la fonction $\varphi(x)$ satisfait, pour tout x réel, l'équation fonctionnelle

$$(3.3) \quad \varphi(x) = \prod_{x=1}^{\infty} \varphi(a_k x),$$

où le produit infini est convergent. Alors, $\lambda = 0$ et $\varphi(x) = \exp [\mu x^{2p}]$.

Démonstration. Pour $p = r = 1$ ce théorème a été établi dans [5]. Donc, en considérant le cas général, on observe que la convergence du produit infini impose la condition

$$(3.4) \quad \lim_{k \rightarrow 0} \varphi(a_k x) = 1.$$

D'autre part, l'existence de la limite (3.2) assure la continuité de la fonction $\varphi(x)$ dans le point $x = 0$. Puis, vu que la convergence de la série (3.1) impose avec nécessité la condition $\lim_{k \rightarrow \infty} a_k = 0$, de (3.4) et (3.2) il en résulte

$$(3.5) \quad \varphi(0) = 1.$$

Il s'ensuit alors de (3.2) que

$$(3.6) \quad \varphi(x) = 1 + \lambda x^r + O(x^{2p}) \quad \text{pour } x \rightarrow 0.$$

Mais, parce que

$$(3.7) \quad \exp [\lambda x^r] = 1 + \lambda x^r + O(x^{2r}) \quad \text{pour } x \rightarrow 0$$

et $2r \geq 2p$, il résulte de (3.6) et (3.7) qu'on peut prendre

$$(3.8) \quad \varphi(x) = \exp[\lambda x^r] + O(x^{2p}) \quad \text{pour } x \rightarrow 0,$$

d'où on tire pour le produit partiel $\prod_{k=1}^n \varphi(a_k x)$ l'expression

$$(3.9) \quad \prod_{k=1}^n \varphi(a_k x) = \exp\left[\lambda x^r \sum_{k=1}^n a_k^r\right] + O(x^{2p}) \quad \text{pour } x \rightarrow 0.$$

La convergence du produit infini de (3.3) impose la convergence de la série $\sum_{k=1}^{\infty} a_k^r$. Par conséquent, en notant

$$\sum_{k=1}^{\infty} a_k^r = S$$

et compte tenu de (3.1), il résulte qu'on a $S > 1$. Maintenant, en calculant la limite de (3.9), pour $n \rightarrow \infty$, on obtient

$$\prod_{k=1}^{\infty} \varphi(a_k x) = \exp[\lambda S x^r] + O(x^{2p}) \quad \text{pour } x \rightarrow 0,$$

ou

$$\varphi(x) = \exp[\lambda S x^r] + O(x^{2p}) \quad \text{pour } x \rightarrow 0.$$

Donc on peut écrire

$$\varphi(x) = 1 + \lambda S x^r + O(x^{2p}) \quad \text{pour } x \rightarrow 0,$$

d'où il s'ensuit que

$$(3.10) \quad \frac{\varphi(x) - 1 - \lambda x^r}{x^{2p}} = \frac{\lambda(S-1)}{x^{2p-r}} + O(1) \quad \text{pour } x \rightarrow 0.$$

Mais, (3.10) est compatible avec (3.2) si et seulement si $\lambda = 0$ et, par conséquent, la condition (3.2) prend la forme

$$(3.11) \quad \lim_{x \rightarrow 0} \frac{\varphi(x) - 1}{x^{2p}} = \mu,$$

d'où on déduit encore

$$(3.12) \quad \lim_{x \rightarrow 0} \frac{\ln \varphi(x)}{x^{2p}} = \mu.$$

Alors, il en résulte de (3.11) qu'il existe les nombres positifs ρ et A , de sorte que

$$|\varphi(x) - 1| < A x^{2p} \quad \text{pour } |x| \leq \rho$$

et, par conséquent, pour $|x| \leq \eta = \min \left\{ \rho, \frac{1}{\sqrt[2p]{A}} \right\}$ on a

$$\sum_{k=1}^{\infty} |\arg \varphi(a_k x)| < \pi.$$

Il en résulte donc de (3.3) la relation

$$(3.13) \quad \ln \varphi(x) = \sum_{k=1}^{\infty} \ln \varphi(a_k x), \quad \text{pour } |x| \leq \rho,$$

qui conduit par des itérations, (pour tout nombre naturel m), à l'égalité

$$\ln \varphi(x) = \sum_{k_1=1}^{\infty} \dots \sum_{k_m=1}^{\infty} \ln \varphi(a_{k_1} a_{k_2} \dots a_{k_m} x).$$

En notant $\max a_k = a < 1$, il en résulte que $\max a_{k_1} a_{k_2} \dots a_{k_m} \leq a^m$ et donc, compte tenu de la relation (3.12), il suit que pour tout $\varepsilon > 0$, il existe un nombre m suffisamment grand de sorte que

$$|\ln \varphi(a_{k_1} a_{k_2} \dots a_{k_m} x) - \mu x^{2p} a_{k_1}^{2p} a_{k_2}^{2p} \dots a_{k_m}^{2p}| \leq \varepsilon a_{k_1}^{2p} a_{k_2}^{2p} \dots a_{k_m}^{2p}$$

pour $|x| \leq \eta$. Par conséquent, les deux dernières relations conduisent à l'inégalité

$$|\ln \varphi(x) - \mu x^{2p}| \leq \varepsilon \quad \text{pour } |x| \leq \eta;$$

mais vu que ε peut être choisi assez petit que l'on veut, il en résulte

$$(3.14) \quad \varphi(x) = \exp [\mu x^{2p}] \quad \text{pour } |x| \leq \eta.$$

Donc, il reste à démontrer que (3.14) est une solution de l'équation (3.3) valable pour tout x réel. Pour ce but, on suppose que $\varphi(x)$ a la forme

$$(3.15) \quad \varphi(x) = \psi(x) \exp [\mu x^{2p}],$$

où obligatoirement $\psi(x) = 1$ pour $|x| \leq \eta$. Alors, en vertu de l'équation (3.3) et de l'égalité (3.1), on obtient

$$(3.16) \quad \psi(x) = \prod_{k=1}^{\infty} \psi(a_k x).$$

On considère maintenant x_0 comme la plus grande valeur de x pour laquelle $\psi(x) = 1$. Alors, on a $x_0 \geq \eta > 0$ et supposons x_0 fini. Il en résulte $\psi(\pm x_0) = 1$, car $\max a_k = a < 1$ pour tout $k \in N$.

Soit encore δ un nombre quelconque satisfaisant la double inégalité $a < \delta < 1$. On obtient $a_k/\delta < a_k/a \leq 1$ et par conséquent, $\psi(\pm a_k x_0)/\delta = 1$ pour tout $k \in N$. Il en résulte de (3.16) la relation $\psi(\pm x_0/\delta) = 1$, qui est

en contradiction avec la définition de x_0 . Par suite, $x_0 = +\infty$ et $\psi(x)=1$ pour tout x réel et de même, la solution (3.14) de l'équation (3.3) est valable pour tout $x \in R$.

En conclusion, la démonstration faite pour le cas $p=r=1$ dans [5] s'étend sans aucune difficulté au cas considéré auparavant qui nous sera utile pour le développement d'une étude sur la fonction caractéristique de la loi normale généralisée.

Reçue le 15 avril 1971

Institut Polytechnique, Jassy,
Chaire de Mathématiques I

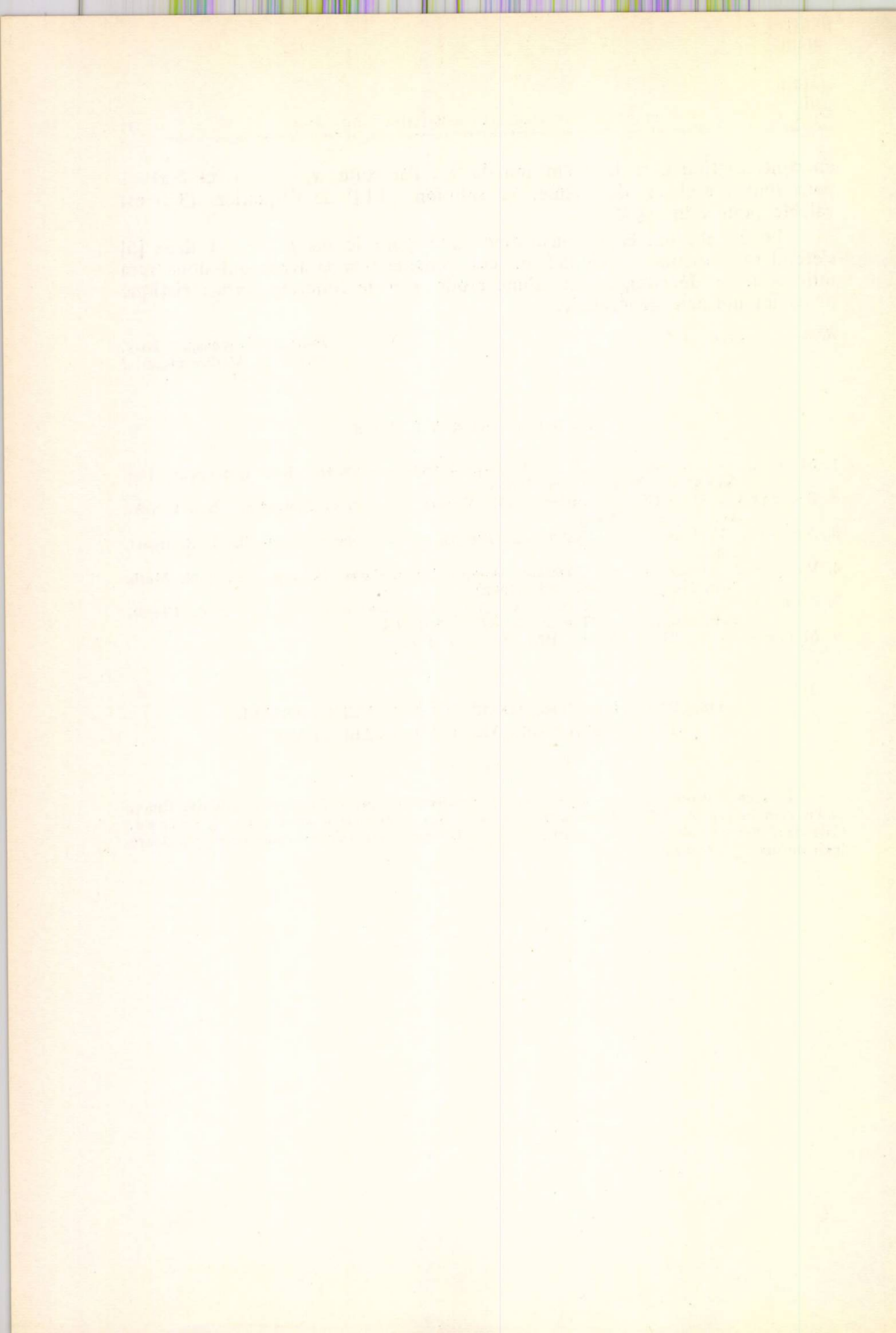
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OBSERVAȚII ASUPRA UNOR ECUAȚII FUNCȚIONALE PRIVIND LEGI NORMALE DE PROBABILITATE

(Rezumat)

În această Notă sînt generalizate unele rezultate din [4] și [5] pentru anumite funcții ce intervin în [1], ele referindu-se la generalizarea legii normale a lui Gauss-Laplace. Cele două teoreme obținute pot contribui la studiul funcțiilor caracteristice corespunzătoare legii normale generalizate.



CRYTÈRES D'EXISTENCE DE LA SOLUTION POUR CERTAINES ÉQUATIONS FONCTIONNELLES

PAR

ADRIAN CORDUNEANU

1. Nous établirons quelques résultats sur l'existence de la solution de certaines équations, en utilisant „la méthode variationnelle“ telle qu'elle est exposée dans [1], [2]. Nous emploierons les notations et les définitions de [1] et supposerons connue la notion de différentielle Fréchet.

Rappelons que si une fonctionnelle réelle définie sur un espace de Banach est différentiable dans un point d'extrême, alors la différentielle dans ce point est identiquement nulle.

X étant un espace de Banach et R l'ensemble des nombres réels, la fonctionnelle $\Phi: X \rightarrow R$ s'appelle convexe si $\forall u_1, u_2 \in X$,

$$(1) \quad \Phi(tu_1 + (1-t)u_2) \leq t\Phi(u_1) + (1-t)\Phi(u_2), \quad \forall t \in [0,1].$$

Désignons, alors qu'elle existe, par $\Phi'(u)$ la dérivée Fréchet de Φ dans le point $u \in X$.

Lemme 1. *Une condition suffisante pour que $\Phi: X \rightarrow R$ soit convexe est qu'elle soit différentiable Fréchet sur X et que sa dérivée $\Phi': X \rightarrow X^*$ soit monotone, c'est-à-dire,*

$$(2) \quad (\Phi'(u) - \Phi'(v), u - v) \geq 0, \quad \forall u, v \in X.$$

Démonstration. u_1 et u_2 étant fixés dans X , considérons la fonction réelle $\varphi(t) = \Phi(tu_1 + (1-t)u_2)$, $t \in R$. En utilisant la notation $u_t = tu_1 + (1-t)u_2$, on obtient

$$(3) \quad \lim_{\Delta t \rightarrow 0} \frac{\varphi(t + \Delta t) - \varphi(t)}{\Delta t} = (\Phi'(u_t), u_1 - u_2),$$

c'est-à-dire φ est dérivable; en prenant $t_1 < t_2$ on déduit que

$$\varphi'(t_2) - \varphi'(t_1) = (\Phi'(u_{t_2}) - \Phi'(u_{t_1}), u_{t_2} - u_{t_1})(t_2 - t_1)^{-1} \geq 0.$$

Donc φ est convexe. En posant la condition que le graphe de φ sur $[0, 1]$ se trouve sous le graphe de la sécante qu'unit ses extrémités, on obtient l'inégalité (1).

Lemme 2. *X étant un espace de Banach réflexif, une condition suffisante pour que $\Phi: X \rightarrow R$ admet minimum en un point $u \in X$ est qu'elle soit continue, convexe et $\lim_{\|u\| \rightarrow \infty} \Phi(u) = +\infty$.*

Ce lemme est un corollaire d'un théorème de Browder, établi en [3].

Dans le présent travail, X sera l'espace $L_2(I)$, où par I nous avons désigné l'intervalle $[0, 1]$ de R ; $(\cdot, \cdot)$ représentera le produit scalaire dans $L_2(I)$.

2. Proposition 1. *Considérons l'équation*

$$(4) \quad f(x, u(x)) + \int_0^1 K(x, y)u(y)dy = u_0(x), \quad x \in I$$

et supposons que :

(i) $f(x, u)$ est mesurable par rapport à $x \in I$, pour tout $u \in R$ fixé, $f'_u(x, u)$ existe sur $I \times R$, $0 \leq f'_u(x, u) \leq M$ et $f(x, 0) \in L_2(I)$;

(ii) $|f_1(x, u)| \leq a_1(x) + b_1 u^2$ sur $I \times R$; $a_1 \in L_1(I)$, $b_1 = \text{const.}$ et $f_1(x, u) = \int_0^u f(x, \tau)d\tau$ sur $I \times R$;

(iii) $f_1(x, u) \geq \alpha u^2 - |\beta(x)| \cdot |u| - |\gamma(x)|$ sur $I \times R$; $\alpha > 0$, $\beta \in L_2(I)$ et $\gamma \in L_1(I)$;

(iv) $K(x, y) = K(y, x)$ sur $I \times R$, l'opérateur K ayant le noyau $K(x, y)$ est continu dans $L_2(I)$ et de plus

$$\mathcal{K}(u) = \frac{1}{2} (Ku, u) = \frac{1}{2} \int_0^1 \int_0^1 K(x, y)u(x)u(y)dx dy \geq 0, \quad \forall u \in L_2(I).$$

Alors l'équation (4) admet au moins une solution $u \in L_2(I)$, $\forall u_0 \in L_2(I)$.

Démonstration. Considérons la fonctionnelle $\Phi(u) = \mathcal{F}(u) + \mathcal{K}(u) -$

$-(u_0, u)$, où par $\mathcal{F}(u)$ nous avons désigné $\int_0^1 f_1(x, u(x))dx$. Φ est continue sur

$L_2(I)$; vraiment, il suffit de montrer seulement que $\mathcal{F}$ est continue. Compte tenu de la condition (ii), il résulte que l'opérateur f_1 défini par la relation $(f_1 u)(x) = f_1(x, u(x))$ est continu sur $L_2(I)$ à valeurs dans $L_1(I)$ ([1], p. 204)

et cela entraîne la continuité de $\mathcal{F}$. On déduit que $\lim_{\|u\| \rightarrow \infty} \Phi(u) = +\infty$, compte tenant des conditions (iii) et (iv). Enfin Φ est différentiable Fréchet en tout point $u \in L_2(I)$, parce que

$$(5) \quad \left\{ \begin{aligned} \Phi(u+h) - \Phi(u) &= \int_0^1 f(x, u(x))h(x)dx + \frac{1}{2} \int_0^1 f'_u(x, u^*(x))h^2(x)dx + \\ &+ (Ku, h) + \frac{1}{2} (Kh, h) - (u_0, h) = (fu + Ku - u_0, h) + \varepsilon(u, h) \|h\|, \\ &\forall h \in L_2(I) \text{ et } \lim_{\|h\| \rightarrow 0} \varepsilon(u, h) = 0. \end{aligned} \right.$$

Par fu nous avons désigné la fonction définie par la relation $(fu)(x) = f(x, u(x))$, $x \in I$; grâce à la condition (i) on déduit $|f(x, u(x))| \leq |f(x, 0)| + M|u|$ sur $I \times R$, donc $fu \in L_2(I)$.

Grâce aux conditions (i) et (iv), la dérivée $\Phi'(u) = fu + Ku - u_0$ est monotone, donc Φ est convexe. Compte tenu des lemmes 1 et 2, il existe $u \in L_2(I)$ avec $\Phi'(u) = fu + Ku - u_0 = 0$ et la démonstration est achevée.

Remarque. Si l'un des opérateurs f et K est strictement monotone, la solution u est unique.

Vraiment, si $v \in L_2(I)$ est une autre solution de l'équation (4), on obtient $fu - fv - K(u - v) = 0$, donc $(fu - fv, u - v) + (K(u - v), u - v) = 0$, et on arrive à la conclusion $v = u$.

Avec les notations introduites on peut énoncer aussi la

Proposition 2. *Admettons que les conditions (ii) et (iii) de la proposition précédente sont remplies et de plus*

(i') $f(x, u)$ est mesurable par rapport à $x \in I$ pour tout $u \in R$ fixé, $f'_u(x, u)$ existe sur $I \times R$, $0 < \alpha_0 \leq f'_u(x, u) \leq M$ et $f(x, 0) \in L_2(I)$;

(iv') $K(x, y) = K(y, x)$ sur $I \times R$, l'opérateur K ayant le noyau $K(x, y)$ est continu dans $L_2(I)$, $\mathcal{K}(u) \geq -\alpha_1(u, u)$, $\forall u \in L_2(I)$, $0 < \alpha_1 \leq \alpha_0$, $\alpha_1 < \alpha$.

Alors l'équation (4) admet au moins une solution $u \in L_1(I)$, $\forall u_0 \in L_2(I)$.

Proposition 3. *Considérons l'équation*

$$(6) \quad u(x) + \int_0^1 K(x, y) f(y, u(y)) dy = u_0(x), \quad x \in I$$

et supposons que la condition (ii) de la proposition (1) est remplie et de plus,

(i'') $f(x, u)$ est mesurable par rapport à $x \in I$ pour tout $u \in R$ fixé, $f'_u(x, u)$ existe sur $I \times R$, $|f'_u(x, u)| \leq M$ et $f(x, 0) \in L_2(I)$;

(iv'') $K(x, y) = K(y, x)$ sur $I \times R$ l'opérateur K ayant le noyau $K(x, y)$ est continu dans $L_2(I)$, $\mathcal{K}(u) \geq \alpha_2(u, u)$, $\forall u \in L_2(I)$, $\alpha_2 > 0$, $b_1 \|K\|^2 < \alpha_2$ et $M \|K\|^2 \leq 2\alpha_2$.

Alors l'équation (6) admet au moins une solution $u \in L_2(I)$, $\forall u_0 \in L_2(I)$.

Démonstration. Considérons la fonctionnelle $\Phi_1(u) = \mathcal{K}(u) + \mathcal{F}(Ku) - (u_0, u)$ qui est continue sur $L_2(I)$ et satisfait à la condition $\lim_{\|u\| \rightarrow \infty} \Phi_1(u) = +\infty$; vraiment $\mathcal{K}(u) \geq \alpha_2(u, u)$ et

$$(7) \quad |\mathcal{F}(Ku)| \leq \int_0^1 |f_1(x, Ku(x))| dx \leq \int_0^1 a_1(x) dx + b_1 \int_0^1 (Ku(x))^2 dx \leq \\ \leq A_1 + b_1 \|K\|^2 \cdot \|u\|^2, \quad \forall u \in L_2(I).$$

Compte tenu des conditions imposées on obtient donc

$$(8) \quad \Phi_1(u) \geq (\alpha_2 - b_1 \|K\|^2) \cdot \|u\|^2 - \|u_0\| \cdot \|u\| - A_1, \quad \forall u \in L_2(I).$$

Φ_1 est différentiable Fréchet en tout point $u \in L_2(I)$, parce que

$$(9) \quad \Phi_1(u+h) - \Phi_1(u) = (Ku + Kf(Ku) - u_0, h) + \varepsilon(u, h) \|h\|,$$

où $\lim_{\|h\| \rightarrow 0} \varepsilon(u, h) = 0$.

La dérivée Φ'_1 est monotone, ce qu'on voit de la relation

$$(10) \quad (\Phi'_1(u) - \Phi'_1(v), u-v) = (K(u-v), u-v) + \\ + (K[f(Ku) - f(Kv)], u-v) \geq (2\alpha_2 - M \|K\|^2) \cdot \|u-v\|^2 \geq 0, \\ \forall u, v \in L_2(I).$$

En appliquant de nouveau les lemmes 1 et 2, on déduit qu'il existe $v \in L_2(I)$ ainsi que $\Phi'_1(v) = 0$; en posant $Kv = u$, on obtient $u + Kfu = u_0$ et la démonstration est achevée.

Proposition 4. Dans le mêmes conditions pour f et K que dans la proposition précédente, l'équation

$$(11) \quad u(x) + f\left(x, \int_0^1 K(x, y)u(y)dy\right) = u_0(x), \quad x \in I$$

admet au moins une solution $u \in L_2(I)$, $\forall u_0 \in L_2(I)$.

Démonstration. Il suffit de prendre $\Phi_2(u) = \mathcal{K}(u) + \mathcal{F}(Ku) - (Ku_0, u)$, d'écrire la relation $\Phi'_2(u) = 0$ sous la forme $K(u + f(Ku) - u_0) = 0$ et de tenir compte que K est inversable, grâce à la condition (iv'').

Remarque. Des résultats analogues à ceux établis dans ce travail ont été obtenus en [4] par une voie un peu différente.

Reçue le 26 février 1970

Institut Polytechnique, Jassy,
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CRITERII DE EXISTENȚĂ A SOLUȚIEI PENTRU UNELE ECUAȚII
FUNCȚIONALE

(Rezumat)

Alegindu-se în mod convenabil anumite funcționale reale Φ , diferențiabile Fréchet, definite pe $L_2(I)$, se scrie ecuația funcțională considerată în forma $\Phi'(u)=0$. Folosind lemele 1 și 2 se obțin condiții suficiente de existență a soluției pentru ecuațiile (4), (6) și (11).



OPTIMIZATION OF CONTROL SYSTEMS DESCRIBED
BY VOLTERRA INTEGRAL EQUATIONS ¹⁾

BY

C. F. LEE

1. During the past decade a great deal of investigation has been made on control systems involving time-delays. A detailed study of control systems governed by delay-differential equations can be found in [1]. However, very few papers are devoted to control systems described by integral equations. In this connection the work of A. Friedman [2] should be mentioned. He investigates in [2] a control system which is governed by a system of quasilinear Volterra integral equations of second kind with convolution kernels, in which control variables appear under the integral signs.

In this note we consider a control system described by linear Volterra integral equations whose forcing terms contain the control variables only.

2. Consider a control system whose state at time t is described by an n -dimensional real-valued vector function $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. Let R be a compact and convex set in the r -dimensional euclidean space E^r . We assume that the control at time t is represented by an r -dimensional real-valued vector function $u(t) = (u_1(t), u_2(t), \dots, u_r(t))$ such that $u(t) \in R$ for any t in a specified time interval.

The state of the system at the initial time $t = t_0$ is denoted by x_0 . It is assumed that $x_0 \in G \subset E^n$, when G is a compact set containing the origin of the space E^n .

Let $[t_1, t_2]$, where $t_0 \leq t_1 \leq t_2$, be a compact time interval. We assume that the control system is governed by a linear Volterra integral equation of the second kind:

$$(1) \quad x(t) = A(t)u(t) + \int_{t_0}^t K(t, \tau)x(\tau)d\tau, \quad t \in [t_0, t_2],$$

¹⁾ The results in this paper were first presented in an M. Sc. thesis submitted to the University of Queensland in 1966 under the supervision of Professor M. N. Öguztöreli.

where $A(t) = \{A_{ij}(t)\}$, ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, r$), is a continuous $n \times r$ matrix, $K(t, \tau) = \{K_{pq}(t, \tau)\}$, ($p = 1, 2, \dots, n$; $q = 1, 2, \dots, n$), is a continuous $n \times n$ matrix.

A continuous vector function $u(t)$ defined for $t \in [t_0, \bar{t}]$, $\bar{t} \in [t_1, t_2]$, with its range in the region R , will be called an admissible control if it satisfies the condition

$$(2) \quad A(t_0)u(t_0) = x_0.$$

The set of all admissible control will be denoted by U . The set U is assumed to be equicontinuous, containing $u(t) = 0$ as an interior point.

The Volterra integral equation (1) for an admissible control $u(t)$ has a unique solution, which will be denoted by $x(t, t_0, x_0; u)$, or $x(t)$.

Note that

$$(3) \quad x(t) = A(t)u(t) + \int_{t_0}^t H(t, \tau)A(\tau)u(\tau)d\tau, \quad t \in [t_0, t_2],$$

where $H(t, \tau) = \{H_{kl}(t, \tau)\}$, ($k = 1, 2, \dots, n$, $l = 1, 2, \dots, n$), is a continuous $n \times n$ matrix, which represents the resolvent of the integral equation (1).

It is assumed that the cost functional $\mathcal{J}(u)$ measuring the performance of the control system is of the form

$$(4) \quad \mathcal{J}(u) = \int_{t_0}^{\bar{t}} g(x(t), u(t); t) dt,$$

where g is a given function which is sufficiently smooth in all of its arguments, $x(t)$ is the solution of equation (1) under the admissible control $u(t)$ defined for $t \in [t_0, \bar{t}]$, which transfers the control system from the initial state $x_0 = x(t_0)$ to the terminal state $\bar{x} = x(\bar{t})$.

Let T be a family of closed sets T_t in E^n , defined for $t \in [t_1, t_2]$. We say that control $u(t)$ transfers the initial state x_0 onto the set T if the solution $x(t, t_0, x_0; u)$ corresponding to the policy $\{x_0, u\}$ satisfies the relations

$$(5) \quad x(\bar{t}, t_0, x_0; u) \in T_{\bar{t}}.$$

In this article we deal with the following optimization problem:

Find an admissible control $\bar{u}(t)$ which transfers the control system from the initial state x_0 onto the set T with a minimal cost, i.e.,

$$(6) \quad \mathcal{J}(\bar{u}) = \min_{u \in U} \int_{t_0}^{\bar{t}} g(x(t), u(t); t) dt.$$

3. Consider the set of points $(t, x, \varphi) \in E^{n+2}$, $t \in [t_0, t_2]$, $x \in E^n$, and $\varphi \in [0, \infty)$. A point $(\tilde{t}, \tilde{x}, \tilde{\varphi}) \in E^{n+2}$ is said to be attainable if there exists an admissible control $\tilde{u}(t)$ such that equation (1) with the initial condition

$$(7) \quad \tilde{x}(t_0) = x_0 \in G$$

has the unique solution

$$(8) \quad \tilde{x}(t) = A(t)\tilde{u}(t) + \int_{t_0}^t H(t, \tau)A(\tau)\tilde{u}(\tau)d\tau$$

for $t \in [t_0, \tilde{t}]$, $\tilde{t} \in [t_1, t_2]$, satisfying

$$(9) \quad \tilde{x}(\tilde{t}) = \tilde{x}$$

and

$$(10) \quad \tilde{\varphi} = \int_{t_0}^{\tilde{t}} g(\tilde{x}(t), \tilde{u}(t); t) dt.$$

The set of all attainable points will be called the attainable set for the control system and will be denoted by A .

We have the following theorems which are extensions of the results due to E. Roxin [4] and L. W. Neustadt [5], see also [3] and [6]:

Theorem 1. *Under the conditions stated in the last section, the attainable set A is compact.*

Theorem 2. *In addition to the conditions stated in Theorem 1, if*

a) *the closed sets T_i are upper semi-continuous with respect to inclusion, and*

b) *there is at least one admissible pair $\{x_0, u(t)\}$ transferring x_0 onto the set T_i ,*

then there is an optimal pair $\{x_0, \bar{u}(t)\}$ such that $\bar{u}(t)$ transfers x_0 onto the set T with minimal cost.

4. By using the first variations of the cost functional (4), we can easily derive the following necessary condition for the optimal control [3]:

$$\frac{\partial g(x, u; t)}{\partial x} A(t) + \frac{\partial g(x, u; t)}{\partial u} + \int_t^{\tilde{t}} \frac{\partial g(x, u; \sigma)}{\partial x} H(\sigma, t) A(t) d\sigma = 0,$$

an integral equation with respect to $u = u(t)$.

5. The above results can be extended to control systems described by Volterra integral equations of the form

$$x(t) = A(t)u(t) + f(t) + \int_{t_0}^t K(t, \sigma)x(\sigma)d\sigma, \quad t \in [t_0, t_2],$$

where $f(t)$ is a given n -dimensional continuous vector function, (cf. [3]).

Received December 28, 1970

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OPTIMIZARE ÎN SISTEMELE DE CONTROL DESCRISE DE ECUAȚII INTEGRALE VOLTERRA

(Rezumat)

Se studiază existența controalelor optimale pentru sistemele de control descrise de ecuații integrale liniare Volterra. Se stabilesc condiții necesare pentru existența controalelor optimale.

A NOTE ON WIENER-HOPF EQUATION ARISING
IN ELASTIC CONTACT PROBLEMS

BY

RIAZ KHADEM ¹⁾

1. Introduction

In a previous study [1], two pairs of simultaneous dual integral equations, with Bessel function kernels, were considered and reduced to a single equation of the Wiener-Hopf type with both the sum and difference kernels present. This equation is also encountered when the kernels are trigonometric. It can be shown that an appropriate transformation causes the coefficient of the sum kernel to vanish, leaving

$$(1) \quad f(x) + \beta \int_0^{\infty} f(t) k(x-t) dt = \lambda c(x), \quad x > 0; \quad k(t) = \frac{\sin t}{\pi t},$$

where β, λ are complex. Equation (1) was solved by Spence [2] in 1967. His solution, however, does not cover complex β, λ , but it can be modified easily to do so. Complex λ is trivial in that it requires multiplication by λ . Some marginal differences, however, occur when β is complex. These differences are pointed out here.

Spence considers Eq. (1) when $c(x)$ is of the form

$$(2) \quad c(x) = P\left(\frac{d}{dx}\right) k(x) + Q\left(\frac{d}{dx}\right) l(x),$$

where P and Q are n^{th} degree polynomials in the operator d/dx , and $l(x) = (1 - \cos x)/(\pi x)$. He produces a solution by first considering the homogeneous equation (with the Q terms absent), and superimposing on that the solution of the Q equation (with the P terms absent). Since the standard Wiener-Hopf technique fails to solve (1), Spence replaces the kernel $k(x)$ by $k(x, \alpha)$ choosing α such that $k(x, \alpha) \rightarrow 0$ as $\alpha \rightarrow 0$, and $K(w, \alpha)$, the Fourier transform of $k(x, \alpha)$, is regular, for finite α , in a strip of finite width enclosing the real axis in the w -plane. A suitable candidate for $K(w, \alpha)$ is [2]

$$(3) \quad K(w, \alpha) = \beta^{-1} \{ \exp [\alpha \Phi(w)] - 1 \}, \quad \Phi(w) = \Phi_+(w) - \Phi_-(w), \quad \exp (\alpha \pi) = 1 + \beta$$

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and

$$(4) \quad \Phi_{\pm}(w) = \frac{1}{2i} \log \frac{w-1 \pm i\alpha}{w+1 \pm i\alpha}.$$

As $\alpha \rightarrow 0$ in (4), $\Phi_{\pm}(w)$ becomes

$$(5) \quad \begin{cases} \Phi_{\pm}(w) = i\theta(w) & |w| > 1, \\ \Phi_{\pm}(w) = i\theta(w) \pm \frac{1}{2}\pi & |w| < 1, \end{cases}$$

where

$$(6) \quad \theta(w) = \frac{1}{2} \log \left| \frac{w+1}{w-1} \right| = \begin{cases} \tanh^{-1} w & |w| < 1, \\ \coth^{-1} w & |w| > 1. \end{cases}$$

Having chosen $K(w, \alpha)$ appropriately, the solution of the Wiener-Hopf equation follows by the standard procedure. Note that the assumption $f(x) = 0$ for $x < 0$ causes the right-hand side of (1) to become $\lambda c(x) + e(x)$, where $e(x) = 0$ for $x > 0$. The procedure is explained in detail by Spence and will not be repeated here.

It will suffice merely to point out the marginal differences encountered in modifying Spence's solution to include complex β and λ and a brief description of Spence's analysis, when necessary, for clarity. The modified solution is given below for the homogeneous and the Q equations. The sum of the two solutions is a solution of (1) where $c(x)$ is defined by (2).

2. The Homogeneous Equation

The modification of Spence's work for the homogeneous equation is straightforward and leads to the following:

$$(7) \quad \left. \begin{matrix} f(x) \\ e(-x) \end{matrix} \right\} = \pm \frac{\lambda}{\pi\beta} \sinh \left(\frac{1}{2} \pi \kappa \right) \int_0^1 (1-w^2)^{-\tau/2} \{ E(w, \tau) \cos(\sigma\theta \pm wx) - \\ - iT(w, \tau) \sin(\sigma\theta \pm wx) \} dw,$$

where

$$(8) \quad \kappa = \sigma + i\tau, \quad \sigma = \frac{1}{\pi} \log |1 + \beta|, \quad \tau = \frac{1}{\pi} \arg(1 + \beta),$$

and

$$(9) \quad \left. \begin{matrix} E(w, \tau) \\ T(w, \tau) \end{matrix} \right\} = (1+w)^{\tau} R(w) \pm (1-w)^{\tau} R(-w).$$

The polynomial $R(w)$ follows from the requirement

$$P(-izw) \exp[\kappa \Phi_+(w)] - R(w) \rightarrow 0 \quad \text{as } |w| \rightarrow \infty.$$

3. The Q Equation

The modification of the Q equation is also simple. The only difficulty one encounters is the evaluation of the integral

$$(10) \quad \Omega(w) = \frac{1}{2\pi i} \int_{-1}^1 \frac{H(t)}{t-w} dt,$$

where

$$(11) \quad H(t) = 2i \operatorname{sgn}(t) \sinh\left(\frac{1}{2} \pi \kappa\right) e^{i\kappa\theta(t)}, \quad |t| < 1.$$

The integral can be evaluated in terms of $I(w, \kappa) \pm I(w, -\kappa)$ and $\mathcal{J}(w, \kappa) \pm \mathcal{J}(w, -\kappa)$, where

$$(12) \quad I(w, \kappa) = \int_0^1 \frac{t e^{i\kappa\theta(t)}}{t^2 - w^2} dt, \quad \mathcal{J}(w, \kappa) = \int_0^1 \frac{e^{i\kappa\theta(t)}}{t^2 - w^2} dt.$$

The solution of the Q equation then becomes

$$(13) \quad \left. \begin{matrix} f(x) \\ e(-x) \end{matrix} \right\} = \pm \frac{2}{\pi} \sinh\left(\frac{1}{2} \pi \kappa\right) \int_0^1 [U(w) \cos \varphi + wV(w) \sin \varphi] dw,$$

$\varphi = \kappa\theta \pm wx$ (the $\pm$ sign corresponds to $f(x)$ and $e(-x)$, respectively), where κ is complex, and

$$(14) \quad U(w) = -\frac{\lambda}{2\beta} \left[\frac{2}{\pi} \sinh\left(\frac{1}{2} \pi \kappa\right) \right] \{B_1(w) \psi(w, \kappa) + iwB_2(w) \chi(w, \kappa)\} - \\ - \frac{\lambda}{2\beta} [S(w) + S(-w)],$$

$$(15) \quad V(w) = -\frac{\lambda i}{2\beta w} \left[\frac{2}{\pi} \sinh\left(\frac{1}{2} \pi \kappa\right) \right] \{B_2(w) \psi(w, \kappa) + iwB_1(w) \chi(w, \kappa)\} + \\ + \frac{\lambda i}{2\beta w} [S(w) - S(-w)].$$

The functions $\psi(w, \kappa)$, $\chi(w, \kappa)$, $B_1(w)$ and $B_2(w)$ in (14) and (15) are defined below

$$(16) \quad \left. \begin{matrix} \psi(w, \kappa) \\ \chi(w, \kappa) \end{matrix} \right\} = \operatorname{cosech}\left(\frac{1}{2} \pi \kappa\right) \int_0^{\pi/2} \left\{ \frac{\sinh(\kappa y)}{\cosh(\kappa y)} \cot y \right\} \frac{dy}{\cos^2 y + w^2 \sin^2 y}$$

and

$$(17) \quad \left. \begin{matrix} B_1(w) \\ B_2(w) \end{matrix} \right\} = Q(iw) \pm Q(-iw).$$

The function $S(w)$ in (14) and (15) is related to $Q(-iw)$ in the same manner $R(w)$ was related to $P(-iw)$ before.

It should be pointed out that Spence's solution for the special case $\beta < -1$, $\lambda = 1$, contains a misprint. The correct expression in

$$(18) \quad f(x) = \frac{1}{\pi(1 + \beta)^{1/2}} \int_0^1 (1 - w^2)^{1/2} [B \cos \varphi - wp_2 \sin \varphi] - \\ - \frac{iA}{(1 - w^2)^{1/2}} [\cos \varphi - iw \sin \varphi] dw,$$

where $\varphi = \sigma\theta + wx$, $A = p_0 + \kappa p_1 + \frac{1}{2} \kappa^2 p_2$, and $B = p_1 + \kappa p_2$; p_0 , p_1 and p_2 are the coefficients of the second degree polynomial P .

Received June 26, 1970

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NOTĂ ASUPRA ECUAȚIEI WIENER-HOPF CE INTERVINE ÎN PROBLEME DE CONTACT ELASTIC

(Rezumat)

Ecuația Wiener-Hopf [1] a fost rezolvată în 1967 de Spence [2], pentru λ și β reali. Se modifică soluția pentru a acoperi și cazul λ , β complecși, subliniind că apar diferențe legate numai de β , care se analizează în detaliu. Se semnalează totodată o eroare strecurată în soluția lui Spence pentru $\beta < -1$, $\lambda = 1$, care se corectează.

ALGEBRAIC BASIS IN COMMUTATIVE QUASIVECTOR SPACES

BY

DAN BORȘ

Let (X, X^0, X_0) be a quasivector space [2] over a field K .

Definitions. The subset $A = \{a_1, \dots, a_n\}$ of X is called linearly dependent if there exists a set of scalars $\{\alpha_1, \dots, \alpha_n\}$ of K such that

$$(1) \quad \sum_{i=1}^n \alpha_i a_i \in X^0$$

and at least one scalar is different from zero. The subset A is called linearly independent if the relation (1) implies $\alpha_i = 0$ for each $i = 1, 2, \dots, n$.

Proposition 1. *If (X, X^0, X_0) is a commutative quasivector space over a field K and $A = \{a_1, \dots, a_n\}$, $A_0 = \{a_1 + 0, a_1, \dots, a_n + 0, a_n\}$ are subsets of X , then the subsets A and A_0 of X are simultaneously linearly dependent or independent.*

Proof. If A is linearly dependent we have

$$(2) \quad \sum_{i=1}^n \alpha_i a_i + \sum_{i=1}^n \alpha_i 0 \cdot a_i = \sum_{i=1}^n \alpha_i (a_i + 0 \cdot a_i) \in X^0 \cap X_0 = \{0\},$$

therefore A_0 is linearly dependent in X_0 . If A is linearly independent, (2) implies $\alpha_i = 0$ for each $i = 1, 2, \dots, n$, therefore A_0 is linearly independent

in X_0 . If A_0 is linearly dependent in X_0 , the relation $\sum_{i=1}^n \alpha_i (a_i + 0 \cdot a_i) = 0$

means $\sum_{i=1}^n \alpha_i a_i + \sum_{i=1}^n \alpha_i 0 \cdot a_i \in X^0$ or $\sum_{i=1}^n \alpha_i a_i \in X^0$, therefore A is linearly de-

pendent. If A_0 is linearly independent in X_0 , the relation $\sum_{i=1}^n \alpha_i (a_i + 0 \cdot a_i) = 0$

means $\sum_{i=1}^n \alpha_i a_i \in X^0$ and it implies $\alpha_i = 0$ for each $i = 1, 2, \dots, n$, therefore A is linearly independent. Q. e. d.

Proposition 2. If (X, X^0, X_0) is a quasivector space over a field K then

- a) $A \cap X^0 = \emptyset$ for each linearly independent subset of X ,
- b) every subset of quasivector subspace X^0 is linearly dependent.

We note that every linearly independent subset of X does not contain θ .

Proposition 3. If (X, X^0, X_0) is a quasivector space over a field K then

- a) every subset of a linearly independent subset of X is again linearly independent,
- b) the elements of a linearly independent subset of X are different,
- c) every superset of a linearly dependent subset of X is again linearly dependent.

Definitions. A vector x of X is called a left (right) linear combination of the elements $a_1, \dots, a_n$ of X if there exists the scalars $\alpha_1, \dots, \alpha_n$ of K such that $\sum_{i=1}^n \alpha_i a_i + x \in X^0$ ($x + \sum_{i=1}^n \alpha_i a_i \in X^0$). The element x of X is called a linear combination of the elements $a_1, \dots, a_n$ of X if it is a left and right linear combination of these elements.

Proposition 4. If (X, X^0, X_0) is a commutative quasivector space over a field K , then the following conditions are equivalent:

- a) the subset $A = \{a_1, \dots, a_n\}$ of X is linearly dependent,
- b) there exists an element of A which is a linear combination of the other elements of A .

Proposition 5. If (X, X^0, X_0) is a commutative quasivector space over a field K , $A = \{a_1, \dots, a_n\}$ is a linearly independent subset of X and if x of X is a linear combination of the elements of A then this linear combination is uniquely determined.

The inverse statement is not true.

Proof. If $x + \sum_{i=1}^n \alpha_i a_i \in X^0$ and $x + \sum_{i=1}^n \beta_i a_i \in X^0$ then $\sum_{i=1}^n (\alpha_i - \beta_i) a_i + \sum_{i=1}^n 0 \cdot a_i \in X^0$ or $\alpha_i = \beta_i$, ($i = 1, 2, \dots, n$). As a counterexample, we know [1] that we have uniquely $(A, x) = (A, \theta) + (\emptyset, x)$ or $(A, x) + (A, \theta) - (\emptyset, x) = (\emptyset, \theta) \in (T \times Y)^0$ but the set $\{(A, \theta), (\emptyset, x)\}$ is linearly dependent because (A, θ) is in $(T \times Y)^0$ (Propositions 2 and 3).

Definition. The subset A of X is called an algebraic basis of X if it is linearly independent and every element of X is a linear combin-

ation of elements of A . These linear combinations are uniquely determined (Proposition 5) and the scalars are called the coordinates of the element in the basis A .

Proposition 6. *If $A = \{a_1, \dots, a_n\}$ is an algebraic basis of a commutative quasivector space (X, X^0, X_0) over a field K and if $x + \sum_{i=1}^n \xi_i a_i \in X^0, y + \sum_{i=1}^n \eta_i a_i \in X^0$ then $x + y + \sum_{i=1}^n (\xi_i + \eta_i) a_i \in X^0$ and $\alpha \cdot x + \sum_{i=1}^n (\alpha \xi_i) a_i \in X^0$.*

Proposition 7. *If the conditions of Proposition 1 are satisfied, then the sets A and A_0 are simultaneously algebraic basis in X and X_0 ; every algebraic basis in X_0 is also a basis in X .*

Proof. If A is linearly independent then also is A_0 . Let be x_0 in X_0 ; there exists x in X such that $x_0 = x + 0 \cdot x$ because $X = X^0 \oplus X_0$; therefore $x = x_0 + 0 \cdot x$ is the following unique linear combination in basis A :

$$x_0 + 0 \cdot x + \sum_{i=1}^n \alpha_i a_i \in X^0 \quad \text{or} \quad x_0 + \sum_{i=1}^n \alpha_i (a_i + 0 \cdot a_i) \in X^0;$$

therefore A_0 is an algebraic basis in X_0 and the coordinates of x_0 in A_0 are the same for x in A . If A_0 is linearly independent then also is A . Let be x in X ; there exists x_0 in X_0 such that $x = x_0 + 0 \cdot x$ because $X = X^0 \oplus X_0$; therefore $x_0 = x + 0 \cdot x$ is the following unique linear combination in basis A_0 :

$$x + 0 \cdot x + \sum_{i=1}^n \alpha_i (a_i + 0 \cdot a_i) \in X^0 \quad \text{or} \quad x + \sum_{i=1}^n \alpha_i a_i \in X^0;$$

therefore A is an algebraic basis in X and the coordinates of x in A are the same for x_0 in A_0 . If A is an algebraic basis in X then A_0 is an algebraic basis in X_0 . Let be x in X ; then we have uniquely

$$x + \sum_{i=1}^n \alpha_i a_i \in X^0 \quad \text{or} \quad x + \sum_{i=1}^n \alpha_i (a_i + 0 \cdot a_i) \in X^0,$$

therefore A_0 is an algebraic basis in X_0 . If A is an algebraic basis in X_0 and x is in X , there exists uniquely x_0 in X_0 such that $x = x_0 + 0 \cdot x$; we have uniquely $x_0 + \sum_{i=1}^n \alpha_i a_i \in X^0$ or $x + \sum_{i=1}^n \alpha_i a_i \in X^0$, therefore A is an algebraic basis in X . Q.e.d.

The Proposition 7 shows that all finite algebraic basis in a commutative quasivector space have the same number of elements; we give the

Definition. The number of elements of an algebraic basis in X is called the dimension of X over K and is denoted by $\dim_K X$.

Theorem. If (X, X^0, X_0) is a commutative quasivector space over a field K then $\dim_K X = \dim_K X_0$.

Received 21 November, 1970

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BAZE ALGEBRICE ÎN SPAȚII CUASIVECTORIALE COMUTATIVE

(Rezumat)

Se arată că toate bazele algebrice finite ale unui spațiu cuasivectorial comutativ [2] (X, X^0, X_0) peste un câmp K au același număr de elemente și că dimensiunea lui X peste K este egală cu dimensiunea lui X_0 peste K .

ON THE FOURIER TRANSFORM OF DISTRIBUTIONS (II)

BY

P. CRĂCIUNĂȘ

1. Introduction

The subject of the present paper is related with various results concerning the Fourier transform of distributions in the sense of Mikusinski.

It is worth to be mentioned here different problems concerning Schwartz's theory of distributions in the frame of Mikusinski's definition which are to be found in the author's dissertation [7]¹⁾.

In one of our previous paper, published in the same *Bulletin* [6], we have defined the fundamental $\mathcal{F}$ -admissible sequences and the tempered distributions. It was showed that if $f(x)$ is tempered, given by the $\mathcal{F}$ -admissible sequence $\{f_n(x)\}$, its Fourier transform $\mathcal{F}f(x)$ is given by the sequence $\{\mathcal{F}f_n(x)\}$, also $\mathcal{F}$ -admissible. Finally it was proved that the set $\mathcal{M}_{\mathcal{F}}$ of so defined distributions is similar with $\mathcal{S}'$ [2], as well as by the Fourier transform of a distribution of $\mathcal{M}_{\mathcal{F}}$ one obtains the same result as in $\mathcal{S}'$. Also some of classical examples were discussed in the sense of our point of view.

2. Properties of the Tempered Distributions

Theorem 1. *If $f(x)$ and $g(y)$ are two tempered distributions defined on R^v respectively, R^u , then the tensorial product $f(x) \otimes g(y)$ is a tempered distribution defined on $R^v \otimes R^u$ and one obtains $\mathcal{F}(f(x) \otimes g(y)) = \mathcal{F}f(x) \otimes \mathcal{F}g(y)$.*

¹⁾ Other papers related to the Fourier transform and to the theorie of distributions was mentioned in the bibliography of [6].

Proof. Indeed, if $f(x) = [f_n(x)]$ and $g(y) = [g_n(y)]$ with $\{f_n(x)\}$ and $\{g_n(y)\}$ fundamental $\mathfrak{F}$ -admissible sequences, the sequence $\{f_n(x) \otimes g_n(y)\}$ will be also fundamental $\mathfrak{F}$ -admissible sequence. Because the $f(x) \otimes g(y)$ is defined by $\{f_n(x) \otimes g_n(y)\}$, it results finally $\mathfrak{F}(f(x) \otimes g(y)) = [\mathfrak{F}(f_n(x) \otimes g_n(y))] = [\mathfrak{F}f_n(x) \otimes \mathfrak{F}g_n(y)] = \mathfrak{F}f(x) \otimes \mathfrak{F}g(y)$.

To define the multiplicative product as well as the convolution it is necessary to introduce two spaces of operators, namely: Q_M as been the spaces of slowly increasing indefinitely differentiable functions and Q'_c , a subspace of $\mathcal{M}'_{\mathfrak{F}}$ formed by the rapidly decreasing distributions.

We will say that a indefinitely differentiable function $f(x)$ belongs to Q_M if for everyone multy index $p \geq 0$ there exists a polynomial $Q_p(x)$ such that $|f^{(p)}(x)| \leq Q_p(x)$. A sequence $\{f_j(x)\}$ of functions Q_M converges to zero in Q_M if for any $p \geq 0$ there exists a polynomial $Q_p(x)$ such that $\{f_j^{(p)}(x)/Q_p(x)\}$ converges to zero in $L^1(R^v)$.

It is easy to verify that Q_M is a vectorial local convex complet space.

Definition 1. We will say that a sequence of indefinitely differentiable functions $\{f_n(x)\}$, belonging to $L^p(R^v)$ as well as all their derivatives is p -fundamental, $1 \leq p \leq \infty$, if there exists an multy index $l \geq 0$ and $F_{n,k}(x)$, $F_k(x)$ belonging to $L^p(R^v)$, $F_{n,k}(x)$ as well as all their derivatives such that

$$f_n(x) = \sum_{k \leq l} F_{n,k}^{(k)}(x) \quad \text{and} \quad F_{n,k}(x) \rightarrow F_k(x) \text{ in } L^p(R^v).$$

We denote with $\mathcal{M}'_{L^p}$ the set of the distributions which contain in their class of equivalence at least a p -fundamental sequence.

Theorem 2. A distribution $f(x)$ belongs to $\mathcal{M}'_{L^p}$ ($1 \leq p \leq \infty$) if and only if:

- i) for every $\alpha(x) \in C_0^\infty(R^v)$, $\alpha(x) * f(x) \in L^p(R^v)$;
- ii) $f(x)$ can be represented as a finite sum of derivatives of functions belonging to $L^p(R^v)$, (the derivatives is taken in the sense of distributions).

Proof. Let $f(x)$ be of $\mathcal{M}'_{L^p}$ and $\{f_n(x)\}$ be a p -fundamental sequence defining $f(x)$. Then we have $\alpha(x) * f(x) = [\alpha(x) * f_n(x)]$ and

$$\begin{aligned} \alpha(x) * f_n(x) &= \alpha(x) * \sum_{k \leq l} F_{n,k}^{(k)}(x) = \sum_{k \leq l} \alpha^{(k)}(x) * F_{n,k}(x) \rightarrow \\ &\rightarrow \sum_{k \leq l} \alpha^{(k)}(x) * F_k(x) = G(x) \text{ in } L^p(R^v). \end{aligned}$$

Hence $\{\alpha(x) * f_n(x)\}$ defines the distribution $G(x)$ which is a function of $L^p(R^v)$.

Now, if for every $\alpha(x) \in C_0^\infty$, $\alpha(x) * f(x) \in L^p(R^v)$ and we take $\alpha(x) = \gamma(x)E(x)$ with $\gamma(x)E(x)$ a solution of the equation $\Delta^k * \gamma(x)E(x) =$

— $\zeta(x) = \delta(x)$, $\zeta(x) \in \mathcal{D}[2]$, where Δ^k is conveniently chosen in order that the product $\gamma(x) E(x) \times f(x)$ can be defined, then one obtains

$$f(x) = \Delta^k \times (\gamma(x) E(x) \times f(x)) - \zeta \times f(x)$$

where $\gamma(x) E(x) \times f(x)$ belongs to $L^p(R^v)$ alike as $\zeta(x) \times f(x)$ and it results that $f(x)$ can be represented as a finite sum of derivatives of functions of $L^p(R^v)$.

If now $f(x) = \sum_{k \leq l} F_k^{(k)}(x)$ with $F_k(x) \in L^p(R^v)$, then let us consider the sequences of indefinitely differentiable functions $\{F_{n,k}(x)\}$ belonging as well as all their derivatives to $L^p(R^v)$, and such that $F_{n,k}(x) \rightarrow F_k(x)$ in $L^p(R^v)$. We will have then

$$f(x) = \left[\sum_{k \leq l} F_{n,k}^{(k)}(x) \right] = \left[\sum_{k \leq l} f_{n,k}(x) \right] = [f'_n(x)]$$

and it results that $\{f'_n(x)\}$ is a p -fundamental sequence, i.e. $f(x) \in \mathcal{M}'_{L^p}$.

The necessity of ii) is obtained and the proof is finished.

In the following we remember the fact that if $f(x) \in \mathcal{D}_{L^p}[2]$ with $1 \leq p \leq 2$, $\mathcal{F}f(x) \in \mathcal{D}_{L^{p'}}$ with $p' = p/(p-1)$, alike as its product with any polynomial. If $f(x) \in \mathcal{D}_L$, then $\mathcal{F}f(x)$ is borned alike as its product with any polynomial, i.e. $\mathcal{F}f(x)$ is a rapidly decreasing function. If now $f(x) \in \mathcal{M}'_L$, then $f(x)$ is a finite sum of derivatives of functions of $L^1(R^v)$ and it results that $\mathcal{F}f(x)$ is the product of a polynomial with a borned function i.e. $\mathcal{F}f(x)$ is a slowly increasing function. Generally if $f(x) \in \mathcal{M}'_{L^p}$ ($1 < p \leq 2$) $\mathcal{F}f(x) \in \mathcal{M}'_{L^{p'}}$.

Definition 2. The distribution $f(x) \in \mathcal{Q}'_c$ if $f(x)$ contains in its class of equivalence a fundamental sequence $\{f_n(x)\}$ such that there exists a finite number of sequences $\{F_{n,l}(x)\}$ $l \leq p$ of indefinite differentiable rapidly decreasing functions and rapidly decreasing functions $F_l(x)$ such that we have $f_n(x) = \sum_{l \leq p} F_{n,l}^{(l)}(x)$ and $\{F_{n,l}(x)\}$ converges to $F_l(x)$ in the space of rapidly decreasing functions [2]. As in [2] we can prove the following theorem:

Theorem 3. A distribution $f(x)$ belongs to $\mathcal{Q}'_c$ if and only if :

- i) for any $\alpha(x) \in C_0^\infty(R^v)$, $f(x) \times \alpha(x)$ is a continuous rapidly decreasing function;
- ii) $f(x)$ can be represented as a finite sum $\sum_{l \leq p} F_l^{(l)}(x)$ where $F_l(x)$ are rapidly decreasing functions.

Proof. It is well known [1] that $\alpha(x) \times f(x)$ is a continuous function. On the other hand, by the majoration $(1 + |x|^2) \leq C(1 + |t|^2)(1 + |x - t|^2)$

it results

$$\begin{aligned}
 |(1 + |x|^2)^{k/2} (\alpha(x) * f(x))| &= |(1 + |x|^2)^{k/2} \left[\sum_{l \leq p} \alpha^{(l)}(x) * F_{n,l}(x) \right]| \leq \\
 &\leq \sum_{l \leq p} |(1 + |x|^2)^{k/2} [\alpha^{(l)}(x) * F_{n,l}(x)]| = \sum_{l \leq p} |(1 + |x|^2)^{k/2} (\alpha^{(l)}(x) * F_l(x))| \leq \\
 &\leq C \sum_{l \leq p} \left| \iint_{\dots \int_{R^v}} (1 + |t|^2)^{k/2} (t) \alpha^{(l)} F_l(x-t) (1 + |x-t|^2)^{k/2} dt \right| = \\
 &= C \sum_{l \leq p} |(1 + |x|^2)^{k/2} \alpha^{(l)}(x) * (1 + |x|^2)^{k/2} F_l(x)|,
 \end{aligned}$$

and one obtains on the straight hand a finite sum of borned functions as being the products of borned function $(1 + |x|^2)^{k/2} F_l(x)$ and function $(1 + |x|^2)^{k/2} \alpha^{(l)}(x)$ which belongs to $C_0^\infty(R^v)$.

Now if we suppose that for any $\alpha(x) \in C_0^\infty(R^v)$, $\alpha(x) * f(x)$ is a continuous rapidly decreasing function, then if we take $\alpha(x) = \gamma(x) E(x)$ satisfying the same conditions as indicated in Theorem 1, then one obtains a representation for $f(x)$ as a finite sum of rapidly decreasing functions, i.e. $f(x) = \sum_{l \leq p} F_{(l)}^{(l)}(x)$ and if $\{F_{n,l}(x)\}$ is for any l a sequence of indefinitely differentiable functions having the limit $F_l(x)$ in the space of rapidly decreasing functions, then if we take $f_n(x) = \sum_{l \leq p} F_{n,l}^{(l)}(x)$ it results that $f(x) \in Q'_c$ and the proof is finished.

We say that a sequence of d'istributionns $\{f_j(x)\}$ belonging to Q'_c converges to zero in Q'_c if there exists a finite number of sequences $F_{j,i}(x)$ ($i \leq p$) of indefinitely differentiable rapidly decreasing functions and of rapidly decreasing functions $F_i(x)$ such that we have $f_j(x) = \sum_{i \leq p} F_{j,i}^{(i)}(x)$, $F_{j,i}(x) \rightarrow F_i(x)$ in the space of rapidly decreasing functions, and moreover $\sum_{i \leq p} F_i^{(i)}(x) = 0$.

Obviously, a distribution with compact support belongs to Q'_c and the space $\mathcal{D}'$ [2] is dense in Q'_c . Analogously, any indefinite differentiable function, rapidly decreasing, can be considered as being a distribution Q'_c and $\mathfrak{S}$ is dense in Q'_c .

Theorem 4. If $\alpha(x) \in Q_M$ and $f(x) = [f_n(x)] \in \mathcal{M}'_{\mathfrak{F}}$, then we can define the product $\alpha(x)f(x) \in \mathcal{M}'_{\mathfrak{F}}$ as being given by the fundamental $\mathfrak{F}$ -admissible sequence $\{\alpha(x)f_n(x)\}$. Moreover if for any $f(x) \in \mathcal{M}'_{\mathfrak{F}}$ there exists the product $\alpha(x)f(x)$ and belongs to $\mathcal{M}'_{\mathfrak{F}}$, then it is necessary that $\alpha(x) \in Q_M$.

Proof. Because $\alpha(x)$ is indefinite differentiable and $\{f_n(x)\}$ is a simple fundamental sequence, it results [1] that $\{\alpha(x)f_n(x)\}$ is also a fun-

damental sequence. In addition, if $f_n(x) = F_n^{(p)}(x)$ and $F_n(x) \xrightarrow{l} F(x)$ (the corresponding is to be found in [6]) then

$$\alpha(x) f(x) = [\alpha(x) f_n(x)] = \alpha(x) F_n^{(p)}(x) = \sum_{l \leq p} (-1)^{|l|} C_p^l (\alpha^{(l)} F_n(x))^{(p-l)}$$

and it results that $\{\alpha(x) f_n(x)\}$ is a fundamental $\mathfrak{F}$ -admissible sequence.

Now if we suppose that $\alpha(x) f(x) \in \mathcal{U} \mathfrak{F}$ for any $f(x) \in \mathcal{M}'_{\mathfrak{F}}$ taking $f(x) = l$ it results $\alpha(x) \in \mathcal{M}_{\mathfrak{F}}$, i.e. $\alpha(x) \in Q_M$.

Theorem 5. If $f(x) \in Q'_c$ and $g(x) \in \mathcal{M}'_{\mathfrak{F}}$, we can define the product $f(x) \times g(x)$ as being given by the sequence $\{f_n(x) \times g_n(x)\}$ where $f(x) = [f_n(x)]$ and $g(x) = [g_n(x)]$. In addition if, for any $g(x) \in \mathcal{M}'_{\mathfrak{F}}$ there exists the product $f(x) \times g(x)$, then with necessity $f(x) \in Q'_c$.

Proof. Indeed, the sequence $\{f_n(x) \times g_n(x)\}$ is a sequence of indefinitely differentiable rapidly decreasing functions. In addition, if $f_n(x) = \sum_{k \leq m} F_{n,k}^{(k)}(x)$ where $\{F_{n,k}(x)\}$ are a finite number of sequences of indefinitely differentiable rapidly decreasing functions converging to the rapidly decreasing functions $F_k(x)$, and $g_n(x) = G_{n,k}^{(p)}(x)$, with $\{G_n(x)\}$ a sequence of indefinitely differentiable slowly increasing functions converging to a slowly increasing function $G(x)$, then we have $f_n(x) \times g_n(x) = \sum_{k \leq m} (F_{n,k}(x) \times$

$\times G_n(x))^{p+k}$ and using the majoration $\frac{1}{1+|x|^2} \leq C \frac{1+|x-t|^2}{1+|t|^2}$, it results

$$\begin{aligned} \left| \frac{1}{(1+|x|^2)^N} G_n(x) \times F_{n,k}(x) \right| &= \left| \iint \dots \int_{R^v} \frac{1}{(1+|x|^2)^N} G_n(t) F_{n,k}(x-t) dt \right| \leq \\ &\leq C \iint \dots \int_v \left| \frac{G_n(t)}{(1+|t|^2)^N} \right| |F_{n,k}(x-t) (1+|x-t|^2)^N| dt. \end{aligned}$$

If N is suitably chosen, such that $G_n(t)/(1+|t|^2)^N$ will be summable it results that $\left| \frac{1}{(1+|x|^2)^N} G_n(x) \times F_{n,k}(x) \right|$ are borned, i.e. $G_n(x) \times F_{n,k}(x)$ are slowly increasing functions.

Now we can substitute the sum of derivative with a only one derivative of a sequence $\{H_n(x)\}$ of infinitely differentiable slowly increasing functions such that we have $f_n(x) \times g_n(x) = H_n^{(l)}(x)$ and $H_n(x) \xrightarrow{l} H(x)$ i.e. $\{f_n(x) \times g_n(x)\}$ is a fundamental $\mathfrak{F}$ -admissible sequence.

Finally if $f(x) * g(x) \in \mathcal{M}'_{\mathcal{F}}$ for any $g(x) \in \mathcal{M}'_{\mathcal{F}}$, then if we suppose that $f(x) \notin Q'_c$, it results that $f(x)$ has a slowly decrease and then $\frac{1}{f(x)}$ is a tempered distribution and it is easy to see that the product $f(x) * \frac{1}{f(x)}$ can not be defined as a distribution of $\mathcal{M}'_{\mathcal{F}}$.

Lastly we can prove the following:

Theorem 6. *The Fourier transform and its inverse change the multiplicative product in convolution and conversely, i.e. if*

1) $f(x) \in \mathcal{M}'_{\mathcal{F}}$ and $\alpha(x) \in Q'_M$ then $\mathcal{F}f(x) \in \mathcal{M}'_{\mathcal{F}}$, $\mathcal{F}\alpha(x) \in Q'_c$ and $\mathcal{F}(\alpha(x)f(x)) = \mathcal{F}\alpha(x) * \mathcal{F}f(x)$;

2) $f(x) \in \mathcal{M}'_{\mathcal{F}}$ and $g(x) \in Q'_c$ then $\mathcal{F}f(x) \in \mathcal{M}'_{\mathcal{F}}$, $\mathcal{F}g(x) \in Q'_M$ and $\mathcal{F}(g(x) * f(x)) = \mathcal{F}g(x) * \mathcal{F}f(x)$.

Proof. Indeed for 1) we have $\mathcal{F}(\alpha(x)f(x)) = [\mathcal{F}(\alpha(x)f_n(x))] = [\mathcal{F}\alpha(x) * \mathcal{F}f_n(x)]$ and $\mathcal{F}\alpha(x) \in Q'_c$ (see [2]).

To prove 2) we have $\mathcal{F}(f(x) * g(x)) = [\mathcal{F}(f_n(x) * g_n(x))] = [\mathcal{F}f_n(x) \cdot \mathcal{F}g_n(x)]$ and $[\mathcal{F}g_n(x)] = \left[\mathcal{F} \left(\sum_{k \leq m} G_{n,k}^{(n)}(x) \right) \right] = \sum_{k \leq m} [P_k(y) \mathcal{F}G_{n,k}(x)] = \sum_{k \leq m} P_k(y) \mathcal{F}G_n(x)$ i.e. $[\mathcal{F}g_n(x)]$ is a slowly increasing function alike as all its derivatives.

Received April 11, 1971

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ASUPRA TRANSFORMATEI FOURIER A DISTRIBUȚIILOR (II)

(Rezumat)

În această continuare a lucrării [6], se studiază proprietățile distribuțiilor temperate și ale transformatei Fourier referitoare la produsul multiplicativ și produsul de convoluție. În acest scop se definesc, în sensul lui Mikusinski [1], distribuțiile rapid descrescătoare Q'_c și spațiile $\mathcal{M}'_{L,p}$ corespunzătoare spațiilor $\mathcal{D}'_{L,p}$ din [2], dintre care $\mathcal{M}'_{L^1}$ intervine în demonstrația teoremei 6 (vezi [2]).

REMARQUES SUR LA DISTANCE ENTRE LES ENSEMBLES ¹⁾

PAR

GH. TOADER

1. Considérons un espace métrique (X, d) muni d'une topologie compatible avec la métrique d et soient deux ensembles $A, B \subset X$. Définissons la distance entre les ensembles A et B , comme d'habitude, par le nombre: $d(A, B) = \inf \{d(a, b); a \in A, b \in B\}$.

Remarque 1. *La distance entre deux ensembles fermés, disjointes d'un espace métrique quelconque peut être zéro.*

En effet, considérons l'exemple suivant: soit l'espace métrique (X, d) , où $X = [-1, 0] \cup (0, 1]$, et $d(a, b) = |a - b|$. En considérant comme sous-espace de l'espace R , il en résulte que les ensembles $A = [-1, 0]$ et $B = (0, 1]$ sont fermés, disjointes et toutefois la distance $d(A, B) = 0$.

Des [1] on peut déduire que si l'un des deux ensembles A et B est compact, la distance $d(A, B) > 0$. Évidemment, cette condition est suffisante, mais, compte tenu de la remarque 1, on pose naturellement le problème de trouver des conditions nécessaires ainsi que la distance considérée ci dessus soit strictement positive.

2. Nous allons chercher à répondre à cette question, dont l'énoncé général est le suivant:

Quelles conditions doivent être imposées à l'espace métrique (X, d) pour qu'il ait la propriété:

(Z) *quels que soient les ensembles $A, B \subset X$, disjointes et fermés dans l'espace (X, d) , la distance $d(A, B) > 0$?*

Considérons un espace métrique (X, d) qui n'est pas complet. Alors il a une suite fondamentale (x_n) qui ne converge pas. Alors l'ensemble $\{x; n = 1, 2, \dots\}$ est infini et donc si nous éliminons de (x_n) les termes qui ont déjà parus nous obtenons encore une suite (y_n) . On voit que les en-

¹⁾ Présentée lors de la session scientifique de l'Institut Polytechnique de Jassy, le 2-4 juillet 1970.

sembles $A = \{y_{2n-1}; n = 1, 2, \dots\}$ et $B = \{y_{2n}; n = 1, 2, \dots\}$ sont disjointes, fermés (ils n'ont pas des points d'accumulation dans l'espace (X, d)) et toutefois $d(A, B) = 0$. Donc l'espace n'a pas la propriété (Z), ce qui démontre le

Théorème 1. *Si l'espace métrique (X, d) a la propriété (Z) il est complet.*

Remarque 2. *La complétude de l'espace (X, d) n'est pas suffisante pour qu'il ait la propriété (Z), comme le démontre l'exemple suivant: dans l'espace R les ensembles $A = \{n; n = 1, 2, \dots\}$ et $B = \{n - 1/n; n = 1, 2, \dots\}$ sont fermés, disjointes, mais $d(A, B) = 0$.*

Soit maintenant un espace (X, d) qui a la propriété:

(S) *quel qu'il soit $x_0 \in X$, $M > 0$ et $\delta > 0$, ils existent $x', x'' \in X$, $x' \neq x''$, ainsi que $d(x_0, x') > M$ et $d(x_0, x'') > M$, mais $d(x', x'') \leq \delta$.*

Notons avec $\bar{S}(x_0, M, \delta, x', x'')$ la proposition „à $x_0 \in X$, $M > 0$ et $\delta > 0$ ils correspondent par (S) $x', x'' \in X$, $x' \neq x''$ ainsi que $d(x_0, x') > M$ et $d(x_0, x'') > M$ mais $d(x', x'') \leq \delta$ “.

Soit l'élément $a \in X$. Alors nous avons successivement $\bar{S}(a, 1, 1/2, a_1, b_1)$, $\bar{S}(a_1, 2, 1/3, a_2, b_2)$, $\bar{S}(a_2, d(a_1, a_2) + 1, 1/4, a_3, b_3)$. Supposons trouvés les éléments a_n, b_n et notons $A_n = \{a_1, a_2, \dots, a_n\}$, $d_n = d(A_n) = \sup \{d(a, b); a, b \in A_n\}$, $M_{n+1} = d_n + 1$ et $\delta_{n+1} = 1/n + 2$. Nous avons alors $\bar{S}(a_n, M_{n+1}, \delta_{n+1}, a_{n+1}, b_{n+1})$.

Ont lieu les propriétés:

- (a) $d(a_{n+1}, a_i) > 1$,
- (b) $d(a_{n+1}, b_i) > \frac{1}{2}$,
- (c) $d(b_{n+1}, b_i) > \frac{1}{2}$, pour $i = 1, 2, \dots, n$.

En effet, si nous supposons $d(a_{n+1}, a_i) \leq 1$ il résulte $d(a_{n+1}, a_n) \leq d(a_{n+1}, a_i) + d(a_i, a_n) \leq 1 + d_n = M_{n+1}$ qui contredit le mode de construction de a_{n+1} . Du même, si nous supposons qu'il n'a pas lieu (b), on déduit: $d(a_{n+1}, a_i) \leq d(a_{n+1}, b_i) + d(b_i, a_i) \leq 1/2 + 1/2 = 1$ qui contredit (a). Et, enfin, si (c) est faux, il résulte que: $d(b_{n+1}, a_n) \leq d(b_{n+1}, b_i) + d(b_i, a_i) + d(a_i, a_n) \leq 1/2 + 1/2 + d_n = M_{n+1}$ qui contredit la manière de construction de b_{n+1} .

Notons: $A = \{a_i; i = 1, 2, \dots\}$ et $B = \{b_i; i = 1, 2, \dots\}$. De (a) et (c) nous déduisons que les ensembles A et B sont fermés, et de (b) il résulte qu'ils sont disjointes. D'autre part, $d(A, B) = 0$. Il en résulte que (X, d) n'a pas la propriété (Z), ce qui établit le

Théorème 2. *Si l'espace métrique (X, d) a la propriété (Z), il satisfait la condition: (S) il existent $x_0 \in X$, $M > 0$ et $\delta > 0$, ainsi que pour tous $x', x'' \in X$, $x' \neq x''$ pour lesquels $d(x_0, x') > M$ et $d(x_0, x'') > M$, en ait $d(x', x'') > \delta$ “.*

Corollaire. Pour que l'espace (X, d) ait la propriété (Z), il est nécessaire que l'ensemble de ses points d'accumulation ait diamètre fini.

Remarque 3. Pour que l'espace (X, d) ait la propriété (Z), il n'est pas nécessaire que la sphère fermée

$$S_M^{\leq}(x_0) = \{x; d(x, x_0) \leq M\}$$

(qui résulte du théorème 2) soit séparable.

En effet, soit l'espace (X, d) où X est un ensemble quelconque indénombrable et $d(a, b) = 1$ pour $a \neq b$ et $d(a, b) = 0$ pour $a = b$. Cet espace a la propriété (Z) mais la sphère $S_1^{\leq}(x_0) = X$ n'est pas séparable car le seul ensemble dense en lui est X et il n'est pas dénombrable.

Reçue le 16 septembre 1970

Institut Polytechnique, Jassy,
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OBSERVAȚII CU PRIVIRE LA DISTANȚA DINTRE MULȚIMI

(Rezumat)

Se observă că distanța dintre două mulțimi închise și disjuncte ale unui spațiu metric oarecare poate să fie zero și se dau câteva condiții necesare pentru ca distanța dintre orice două mulțimi închise și disjuncte ale unui spațiu metric să fie strict pozitivă.

THE VECTOR PRODUCT IN FINITE DIMENSIONAL EUCLIDEAN SPACES

BY

ZORAN VRCELJ

1. It is of interest to quote the statement formulated as an exercise in [1] and treated also in [2]: *Show that the vector $\text{grad } f \times \text{grad } g$ corresponding to the scalar fields f and g is solenoidal.*

In this paper we generalize this result to n -dimensional Euclidean spaces (E^n) and suggest its application.

2. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the orthonormal set of base vectors and take a set of $n-1$ vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n-1)} \in E^n$. The scalar product we denote by $\langle, \rangle$.

Definition 1. The vector $\mathbf{z} \in E^n$, defined by the equality

$$\langle \mathbf{w}, \mathbf{z} \rangle = \begin{vmatrix} w_1 & w_2 & \dots & w_n \\ v_1^{(1)} & v_2^{(1)} & & v_n^{(1)} \\ \cdot & & & \\ \cdot & & & \\ v_1^{(n-1)} & v_2^{(n-1)} & & v_n^{(n-1)} \end{vmatrix}, \quad (\mathbf{w} \in E^n),$$

is denoted by $\mathbf{v}^{(1)} \times \mathbf{v}^{(2)} \times \dots \times \mathbf{v}^{(n-1)}$ and called the vector product of vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n-1)}$.

We note that so defined vector product differs from that in [3] only by the factor $(-1)^{n-1}$. This vector product is unique.

It is apparent from the definition that the vector product of given $n-1$ vectors can be represented by the symbolic determinant

$$\mathbf{v}^{(1)} \times \mathbf{v}^{(2)} \times \dots \times \mathbf{v}^{(n-1)} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ v_1^{(1)} & v_2^{(1)} & & v_n^{(1)} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ v_1^{(n-1)} & v_2^{(n-1)} & & v_n^{(n-1)} \end{vmatrix}.$$

Let $\mathbf{F} = F^1 \mathbf{e}_1 + F^2 \mathbf{e}_2 + \dots + F^n \mathbf{e}_n$ be a vector field, where the components F^i are given with respect to the orthonormal basis $(\mathbf{e}_i)$. Let $f^i, (i = 1, 2, \dots, n)$ be scalar fields. We shall suppose that

$$F^1, F^2, \dots, F^n \in C^1(G), \quad f^{(1)}, f^{(2)}, \dots, f^{(n)} \in C^2(G),$$

where G is some region in E^n .

The vector fields $\text{grad } f^{(i)}$ are defined by the formulae

$$\text{grad } f^{(i)} = \frac{\partial f^{(i)}}{\partial x^1} \mathbf{e}_1 + \frac{\partial f^{(i)}}{\partial x^2} \mathbf{e}_2 + \dots + \frac{\partial f^{(i)}}{\partial x^n} \mathbf{e}_n, \quad (i = 1, \dots, n).$$

The scalar field $\text{div } \mathbf{F}$ is defined by the formula

$$\text{div } \mathbf{F} = \frac{\partial F^1}{\partial x^1} + \frac{\partial F^2}{\partial x^2} + \dots + \frac{\partial F^n}{\partial x^n}.$$

We introduce the usual notation for the Jacobian:

$$\mathcal{J} = \begin{vmatrix} \frac{\partial f^{(1)}}{\partial x^1} & \frac{\partial f^{(1)}}{\partial x^2} & \dots & \frac{\partial f^{(1)}}{\partial x^n} \\ \frac{\partial f^{(2)}}{\partial x^1} & \frac{\partial f^{(2)}}{\partial x^2} & & \frac{\partial f^{(2)}}{\partial x^n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{\partial f^{(n)}}{\partial x^1} & \frac{\partial f^{(n)}}{\partial x^2} & \dots & \frac{\partial f^{(n)}}{\partial x^n} \end{vmatrix}.$$

By analogy with the three-dimensional case, we adopt the following definitions supposing that a given vector field satisfies certain conditions in some region in E^n .

Definition 2. A vector field is said to be potential if and only if it can be represented by the gradient of a scalar field.

Definition 3. A vector field is said to be solenoidal if and only if the divergence of it equals zero.

3. Now the principal result may be formulated.

Theorem. A vector field in E^n is solenoidal if and only if it can be represented by the vector product of $n-1$ potential fields in the same space.

Proof. Jacobi's theorem (Crelles Journal 27), cited in [2] and [4], immediately implies the following two statements.

If the components of $\mathbf{F} \in E^n$ are cofactors of $\mathcal{J}$ of the elements of the first row, then $\operatorname{div} \mathbf{F} = 0$.

Inversely. If $\mathbf{F} \in E^n$ fulfils the condition $\operatorname{div} \mathbf{F} = 0$, then the equality $\langle \operatorname{grad} f^{(1)}, \mathbf{F} \rangle = \mathcal{J}$ holds, i.e. components of $\mathbf{F}$ are cofactors of $\mathcal{J}$ of the elements of the first row.

To the set of n arbitrary scalar fields $f^{(1)}, f^{(2)}, \dots, f^{(n)}$ of class $C^2(G)$ corresponds the set of n potential vector fields $\mathbf{w} = \operatorname{grad} f^{(1)}$, $\mathbf{v}^{(i)} = \operatorname{grad} f^{(i+1)}$, ($i=1, 2, \dots, n-1$), of class $C^1(G)$.

Let us suppose that a vector field $\mathbf{F}$ of class $C^1(G)$ can be represented by the vector product of $n-1$ potential fields $\mathbf{v}^{(i)}$ of the same class, i.e. in the form

$$\mathbf{F} = \operatorname{grad} f^{(2)} \times \operatorname{grad} f^{(3)} \times \dots \times \operatorname{grad} f^{(n)}.$$

But in virtue of Definition 1 the components of $\mathbf{F}$ are cofactors of $\mathcal{J}$ of the elements of the first row, and by the first statement follows $\operatorname{div} \mathbf{F} = 0$ in G .

Inversely. Let us suppose that a vector field $\mathbf{F}$ of class $C^1(G)$ is solenoidal in G , i.e. $\operatorname{div} \mathbf{F} = 0$ in G . Then according to the second statement the components of $\mathbf{F}$ are cofactors of $\mathcal{J}$ of the elements of the first row and in virtue of Definition 1 the required representation follows.

Thus the established equivalence is valid.

We note the following simple consequences.

1. If $f^{(2)}, f^{(3)}, \dots, f^{(n)}$ are arbitrary scalar fields of class $C^2(G)$, then the vector field $\operatorname{grad} f^{(2)} \times \operatorname{grad} f^{(3)} \times \dots \times \operatorname{grad} f^{(n)}$ is solenoidal in G .

2. Any solenoidal vector field $\mathbf{F}$ of class $C^1(G)$ can be represented by the symbolic determinant

$$\mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ \frac{\partial f^{(2)}}{\partial x^1} & \frac{\partial f^{(2)}}{\partial x^2} & \dots & \frac{\partial f^{(2)}}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^{(n)}}{\partial x^1} & \frac{\partial f^{(n)}}{\partial x^2} & \dots & \frac{\partial f^{(n)}}{\partial x^n} \end{vmatrix},$$

so that the scalar fields $f^{(2)}, f^{(3)}, \dots, f^{(n)} \in C^2(G)$ satisfy the system of partial differential equations $K_i^1 = F^i$, ($i = 1, 2, \dots, n$), where K_i^1 are the cofactors of that determinant of the elements of the first row.

3. The Jacobian of the functional transformation $y^i = f^{(i)}(x^1, x^2, \dots, x^n)$, ($i = 1, 2, \dots, n$), can be written as the scalar product

$$\langle \text{grad } f^{(1)}, \text{grad } f^{(2)} \times \text{grad } f^{(3)} \times \dots \times \text{grad } f^{(n)} \rangle,$$

whose modulus is independent of superscripts permutation.

4. *Application to Electrodynamics.* The magnetic field $\mathbf{H}$ in three-dimensional Euclidean space is solenoidal, i.e. $\text{div } \mathbf{H} = 0$, and consequently it may be derived from a vector potential $\mathbf{A}$ via the relation $\mathbf{H} = \text{curl } \mathbf{A}$.

According to Theorem, the magnetic field $\mathbf{H}$ may be also derived from two scalar fields, $f^{(2)}$ and $f^{(3)}$, via the relation $\mathbf{H} = \text{grad } f^{(2)} \times \text{grad } f^{(3)}$.

Since a scalar potential $f^{(1)}$ exists such that $\mathbf{E} + \mu_0(\partial \mathbf{A} / \partial t) = -\text{grad } f^{(1)}$, where $\mathbf{E}$ is the electric field, a rather surprising result follows: *Any electromagnetic field can be derived from three scalar potentials.*

Acknowledgment. The author wishes to thank Prof. S. Fempl for helpful suggestions and discussions.

Received January 29, 1971

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PRODUSUL VECTORIAL ÎN SPAȚII EUCLIDIENE FINIT DIMENSIONALE

(Rezumat)

Se generalizează pentru un spațiu euclidian n -dimensional faptul cunoscut că vectorul $\text{grad } f \times \text{grad } g$ corespunzător cimpurilor scalare f și g este solenoidal. Ca aplicație în electrotehnică se deduce că orice cimp electromagnetic poate fi derivat din trei potențiale scalare.

THE VARIATIONAL PRINCIPLE OF THE OPTIMAL
PARAMETRICAL DIVISION IN THE FINITE ELEMENT METHOD

BY

VLADIMÍR KOLÁŘ

The finite element method [4], [15], [16], [18] was outlined as a variational method and developed for structural analysis.

The technical conception was not generally interpretable. Recently the mathematical foundation appeared in papers and books [1], [7] ... [10], [19] ... [21]. The finite element method is considered to be a generalized Ritz method with piecewise polynomial basic functions, i.e. a variational method.

The variational principles of mechanics are well known: the classical principles by Lagrange and Castigliano, the new principles by Hu-Washizu [6], [17], Reissner [13] and Rüdiger [14] for continuous fields in the elastic body Ω without dislocations, and the most recent principles by Prager [12] and Bufler [2], [3] with dislocations. The Bufler principle seems to be the most general one when considering an unvariable (given) body Ω with an unvariable boundary Γ , which can of course become deformed. The so called inverse variational principle by Horák [5] takes in account the variation δI^* of the boundary Γ^* . Starting from these principles [2], [3], [5] we can solve some new problems of mechanics in which the body and its boundary are unknown and our task is to find them following the given statical, geometrical or physical conditions, e.g. stress or strain distribution et cet.

In this paper we will consider another generalization of variational principles, which may be useful for the finite element method. We will start from the simplest case. It is the classical technical form of the finite element method considering displacement models of the given body Ω with N deformation parameters $\Delta = [\Delta_1, \Delta_2, \dots, \Delta_N]^T$, (N, N) stiffness matrix K and N external load parameters $F = [F_1, F_2, \dots, F_N]^T$, see [7], [9], [14], [17].

The total potential energy Π of the loaded body Ω is given by the formula

$$(1) \quad \Pi(\Delta) = \frac{1}{2} \Delta^T K \Delta - F^T \Delta$$

and from the Lagrange variational principle

$$(2) \quad \Pi = \min,$$

i.e. $\partial\Pi/\partial\Delta = 0$, we obtain the linear equation set

$$(3) \quad K\Delta = F.$$

The results of this method depend very much on the chosen division $\{\Omega_k\}$ of the body Ω into elements Ω_k , considering the cases with the same number of unknown parameters. Not only the element form but also the density of division in certain regions is of great significance [7], [9], [10], [17], e.g. near the concentrated loads, holes etc. The optimal division $\{\Omega_k\}$ of the body $\Omega = \sum_k \Omega_k$ with the given number of parameters Δ was looked for only by intuition or comparison of a few examples.

We wish to present a new variational principle which makes the optimal division $\{\Omega_k\}$ possible first in the sense of the minimal total potential energy (2), later we will generalise the problem.

Let the division $\{\Omega_k\}$ of the body Ω be variable and let it depend on n division parameters $\beta = [\beta_1, \beta_2, \dots, \beta_n]^T$ by a given and unvariable number N of parameters Δ . Thus we have the n -parametrical division $\{\Omega_k\}$, where Ω_k ($k = 1, 2, \dots, p$) are the finite elements whose forms depend on β . The whole solution $u = V\Delta$ (u —displacement components, V —influence on parametrical functions) and $\sigma = S\Delta$ (σ —stress tensor, S —stress matrix) depends also on β . The stiffness matrix K and the load vector F in (1) are not constant but they represent the functions $K(\beta)$, $F(\beta)$ of β and the deformation parameters $\Delta(\beta)$ are the functions of β too. Then the variational principle (2) has the form:

$$(4) \quad \Pi(\Delta, \beta) = \frac{1}{2} \Delta^T(\beta) K(\beta) \Delta(\beta) - F^T(\beta) \Delta(\beta) = \min$$

and the $(N + n)$ conditions:

$$(5) \quad \frac{\partial\Pi}{\partial\Delta} = 0, \quad \frac{\partial\Pi}{\partial\beta} = 0,$$

are as follows:

$$(6) \quad N \text{ equations:} \quad K(\beta) \Delta(\beta) = F(\beta),$$

$$(7) \quad n \text{ equations:} \quad \frac{\partial\Pi(\Delta, \beta)}{\partial\beta} = 0.$$

The N equations (6) are linear in Δ , but in any case nonlinear in β ; their form is relatively simple and similar to the form (3). The n -equations (7) are more complicated. For programming we have used the tensor form, where the indices β denote the derivation $\partial/\partial\beta = [\partial/\partial\beta_1, \partial/\partial\beta_2, \dots, \partial/\partial\beta_n]^T$ and the repetition of any index in one term signifies the summation in relation to the repeated index ($i, j = 1, 2, \dots, N$):

$$(8) \quad \Pi(\Delta_i, \beta) = \frac{1}{2} K_{ij}(\beta) \Delta_i(\beta) \Delta_j(\beta) - F_i(\beta) \Delta_i(\beta),$$

$$(9) \quad \Pi_{, \beta} = \frac{1}{2} \{K_{ij, \beta} \Delta_i \Delta_j + K_{ij} [\Delta_i \Delta_{j, \beta} + \Delta_j \Delta_{i, \beta}]\} - \\ - [F_{i, \beta} \Delta_i - F_i \Delta_{i, \beta}] = 0.$$

Thus we have obtained the following set of n equations:

$$(10) \quad \Pi_{, \beta_1} = 0, \quad \Pi_{, \beta_2} = 0, \quad \dots, \quad \Pi_{, \beta_n} = 0.$$

In the following we will present only the simplest problem of the optimal n -parametrical division $\{\Omega_k\}$, where only the n parameters $\beta_1, \beta_2, \dots, \beta_N$ are unknown. The two presumptions of this problem are as follows:

1. In the relation $u = V\Delta$ we take the exact influence functions of F as parametrical (shape) functions V . Generally of course they are not polynomials. Further we apply only the compatible finite elements. Then the displacement parameters Δ due to any possible division $\{\Omega_k\}$ of the body Ω can be proved to be fully exact. The solution $u = V\Delta$ need not be exact; it depends on the element load system. The difference between the solution u and the exact solution is given by special primary functions which are generally unknown.

2. Let us know the exact solution u (or at least Δ) for the given loaded body Ω . Thus we know the exact values of Δ when arbitrarily dividing the body Ω into elements Ω_k and we know the functions $\Delta_i(\beta)$ and their derivatives $\Delta_{i, \beta}(\beta)$ too.

Then we can solve the set of n equations (9) for the n unknown division parameters $\beta = [\beta_1, \beta_2, \dots, \beta_N]^T$, because we know the functions $\Delta_i(\beta), \Delta_{i, \beta}(\beta)$. The geometrical and statical functions $K_{ij}(\beta), F_i(\beta)$ and their derivatives $K_{ij, \beta}(\beta), F_{i, \beta}(\beta)$ can be calculated rather easily in all cases, at least numerically, and we need not know any solution for this purpose. The N equations (6) are omitted and are of no significance for this simple problem which can be defined shortly as follows: We have to find a division $\{\Omega_k\}$ attached to the known solution u in the body Ω , which leads to $\Pi = \min$. We take in account only n -parametrical divisions $\{\Omega_k\}$ and a constant (invariable) number N of deformation parameters Δ .

The above problem, only a theoretical one, can be also of practical significance because the results of it can be approximatively used even then when no u results are known. The above consideration reduced the $(N + n)$ equations (6), (7) to the n equations (10), i. e. to the simplest task, when the exact solution of the problem is known and we only look for the optimal division with a given number $N(n)$ of deformation (division) parameters. But we can formulate the problem also more generally, starting from the more general variational principles [2], [3], which may be or may not be of mechanical character [11].

Let us have an arbitrary functional $P(w)$ which is to be minimized, where w is one or more unknown functions of one or more variables in a region Ω with the boundary Γ :

$$(11) \quad P(w) = \min.$$

Let us solve this problem approximately and take in account only the N -parametrical set of functions $w = Wq$ which may be e.g. polynomial in subregions Ω_k (finite elements Ω_k of Ω). The nature of the N parameters

$$(12) \quad q = [q_1, q_2, \dots, q_N]^T$$

may be technical, physical or only mathematical one. Let us introduce the variable division $\{\Omega_k\}$ of the region Ω with some standard condition pertaining to the given problem. The variation $\delta\{\Omega_k\}$ must be n -parametrical, i.e. it depends only on n division parameters

$$(13) \quad \beta = [\beta_1, \beta_2, \dots, \beta_n]^T,$$

which commonly are of geometrical nature (division density factors et cet).

Then the properties of all elements Ω_k and also of the whole N -parametrical model $\{\Omega_k\}$ of the region Ω depend on β . The solution $w = Wq$ depends also on β . The problem (11) is reduced to the $(N + n)$ parametrical problem

$$(14) \quad P[q(\beta)] = \min.$$

The appropriate equation set

$$(15) \quad \delta_{q, \beta} P = \min$$

is always nonlinear and mostly very complicated. Generally we cannot formulate a simpler problem in an analogous way to the above problem (10), because no influence principle is valid for the relation $w = Wq$ and we cannot calculate using the exact values q even if we knew them.

In this paper we did not present a closed theory. We wanted only to show some problems whose theoretical solution is perhaps impossible and we must solve them by a search. The criterion of $\Pi = \min$ or $P = \min$ seems to be good for this purpose. We hope that we shall be able to publish some results in another paper.

Received December 11, 1970

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UN PRINCIPIU VARIATIONAL PENTRU DIVIZAREA PARAMETRICĂ OPTIMĂ ÎN METODA ELEMENTELOR FINITE

(Rezumat)

Se cercetează problema divizării optime a unui domediu Ω în elementele Ω_k . Se definește așa numita divizare n -parametrică și se utilizează drept principiu variațional, principiul lui Lagrange $\Pi = \min.$, reducînd studiul cazului general la unul mai simplu.

SUR UNE NOUVELLE PRÉSENTATION GRAPHIQUE DE LA THÉORIE DES VECTEURS GLISSANTS

II. RÉDUCTION DES SYSTÈMES DE VECTEURS GLISSANTS

PAR

C. HUIU, GABRIELA NEAGU et N. IRIMICIUC

Introduction

Les caractéristiques vectorielles plane-axielles introduites par les auteurs dans la Note [6], permettront, comme on verra dans cette Note, d'effectuer graphiquement la réduction des systèmes de vecteurs glissants.

I. Réduction des systèmes spaciels de vecteurs glissants

Les quatre cas connus de réduction des systèmes spaciels de vecteurs glissants seront reconnus par les caractéristiques suivantes:

I.1. *Le système réductible à zéro*, caractérisé par les conditions:

$$(I.1) \quad \bar{v}_{II} = \bar{v}_{\Delta} = 0, \quad \bar{m}_{II} = \bar{m} = 0,$$

sera reconnu graphiquement par: a) la clôture de la ligne vectorielle des vecteurs $\bar{v}_{\mu_{\Delta}}$; b) la clôture du polygone vectoriel des vecteurs $\bar{v}_{\mu_{II}}$; c) la clôture du polygone funiculaire correspondant aux vecteurs $\bar{v}_{\mu_{II}}$; d) la clôture du polygone vectoriel des vecteurs $\bar{m}_{\mu_{II}}$.

I.2. *Le système réductible à un couple* est caractérisé par les conditions:

$$(I.2) \quad \bar{v}_{II} = \bar{v}_{\Delta} = 0, \quad \bar{m}_{\Delta} \neq 0, \quad \bar{m}_{II} \neq 0,$$

qui ont les traductions graphiques suivantes: a) la clôture de la ligne vectorielle des vecteurs $\bar{v}_{\mu_{\Delta}}$; b) la clôture du polygone vectoriel des vecteurs $\bar{v}_{\mu_{II}}$; c) l'ouverture du polygone funiculaire correspondant aux vecteurs $\bar{v}_{\mu_{II}}$; d) l'ouverture du polygone vectoriel de vecteurs $\bar{m}_{\mu_{II}}$.

Le moment du couple résultant :

$$(I.3) \quad \bar{M}_{\text{coup}} = \bar{m}_{\Delta} + \bar{m}_{\Pi},$$

aura la composante $\bar{m}_{\Delta}$ déterminée comme dans la fig. 1, cette composante s'annulant si le polygone funiculaire est fermé.

I.3. Le système réductible à un vecteur unique, caractérisé par les conditions :

$$(I.4) \quad \begin{cases} \bar{v}_{\Delta} \neq 0, \quad \bar{v}_{\Pi} \neq 0 & (\text{ou, à moins, l'une d'elles}); \\ \bar{m}_{\Delta} = \bar{M}(\bar{v}_{\Pi}) \neq 0, \quad \bar{m}_{\Pi} = \bar{M}(\bar{v}_{\Delta}) \neq 0 & (\text{ou, à moins, l'une d'elles}); \\ \bar{m}_{\Delta} \bar{v}_{\Delta} + \bar{v}_{\Pi} \bar{m}_{\Pi} = |\bar{v}_{\Delta}| |\bar{v}_{\Pi}| (a' - a_0 \cos \varphi) = 0 & (\text{fig. 2}), \end{cases}$$

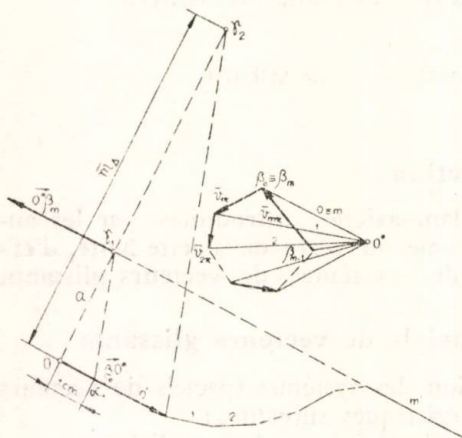


Fig. 1

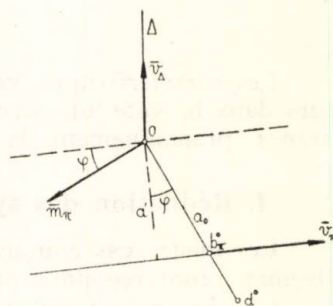


Fig. 2

peut être reconnu par les conditions graphiques: a) l'ouverture de la ligne vectorielle des vecteurs $\bar{v}_{\mu_{\Delta}}$; b) l'ouverture du polygone vectoriel des vecteurs $\bar{v}_{\mu_{\Pi}}$; c) les côtés extrêmes du polygone funiculaire concourants; d) l'ouverture du polygone vectoriel des vecteurs $\bar{m}_{\mu_{\Pi}}$; e) la coïncidence des points d_0 et b_{Π}^0 , (fig. 2), [6], $d_0 \equiv b_{\Pi}^0$ étant le point d'intersection du support de la résultante unique avec le plan Π .

Si la ligne vectorielle des vecteurs $\bar{v}_{\mu_{\Delta}}$ est fermée, alors, en vertu de la deuxième condition (I.4), résulte aussi la clôture du polygone vectoriel des vecteurs $\bar{m}_{\mu_{\Pi}}$ et la résultante unique équivalente au système sera $\bar{v}_{\Pi}$.

Si le polygone vectoriel des vecteurs $\bar{v}_{\mu_{\Pi}}$ est fermé, alors, de la deuxième des conditions (I.4) résulte aussi la clôture du polygone funicu-

laire et le système est réductible au vecteurs $\bar{v}_\Delta$, appliqué en d_0 , point déterminé à l'aide du moment $\bar{m}_\Pi$.

I.4. *Le système réductible à un torseur proprement dit.* Les caractéristiques de ce système :

$$(I.5) \quad \begin{cases} \bar{v}_\Delta \neq 0, & \bar{v}_\Pi \neq 0 \quad (\text{ou, à moins, l'une d'elles}); \\ \bar{m}_\Delta \neq 0, & \bar{m}_\Pi \neq 0 \quad (\text{ou, à moins, l'une d'elles}); \\ \bar{v}_\Delta \bar{m}_\Delta + \bar{v}_\Pi \bar{m}_\Pi \neq 0, \end{cases}$$

se traduisent par les conditions graphiques suivantes: a) l'ouverture de la ligne vectorielle des vecteurs $\bar{v}_{\mu_\Delta}$; b) l'ouverture du polygone vectoriel des vecteurs $\bar{v}_{\mu_\Pi}$; c) l'ouverture du polygone funiculaire correspondant aux vecteurs $\bar{v}_{\mu_\Pi}$; d) l'ouverture du polygone vectoriel des vecteurs $\bar{m}_{\mu_\Pi}$; e) position extérieure du point d_0 , par rapport à la droite Δ_Π , le support du vecteur $\bar{v}_\Pi$.

II. Réduction des systèmes de vecteurs glissants parallèles

On désigne par :

$$0 \leq \psi = \angle(\bar{u}, \bar{u}_\Pi) \leq \frac{\pi}{2},$$

l'angle formé par les supports des vecteurs avec le plan Π , $\bar{u}$ étant le vecteur unitaire commun des vecteurs $\bar{v}_\mu$ et $\bar{u}_\Pi$, le vecteur unitaire commun des vecteurs $\bar{v}_{\mu_\Pi}$, ce qui permet d'écrire :

$$(II.1) \quad \bar{v}_{\mu_\Pi} = v_{\mu_\Pi} \bar{u}_\Pi, \quad \bar{v}_{\mu_\Delta} = v_{\mu_\Pi} \operatorname{tg} \psi \bar{u}_\Delta,$$

$\bar{u}_\Delta$ étant le vecteur unitaire de l'axe Δ [6].

Un vecteur, quelconque, du système est donc déterminé par les vecteurs :

$$(II.2) \quad \bar{v}_\mu \left\{ \begin{array}{l} \bar{v}_{\mu_\Pi} \\ \bar{m}_{\mu_\Pi} \end{array} \right.,$$

les constructions graphiques pour la réduction du système étant ainsi réduites à la construction des polygones vectoriel et funiculaire des vecteurs $\bar{v}_{\mu_\Pi}$ et à la construction du polygone vectoriel des vecteurs $\bar{m}_{\mu_\Pi}$.

Les trois cas de réduction qui peuvent être rencontrés dans cette situation seront reconnus comme on montrera plus bas :

ce qui réduit les constructions graphiques à la construction de la ligne des vecteurs $\bar{v}_{\mu\Delta}$ et au polygone des vecteurs $m_{\mu\Pi}$.

Pour $\psi = 0$ les vecteurs $\bar{v}_{\mu}$ seront déterminés à l'aide des projections plane-axielles de leurs points d'applications et par des vecteurs $\bar{v}_{\mu}$ situés dans le plan Π (fig. 3), chaque vecteur étant remplacé par un ensemble formé par le vecteur $v_{\mu\Pi}$ placé sur l'axe Δ et par un couple, dont le moment est orienté au long de l'axe Δ .

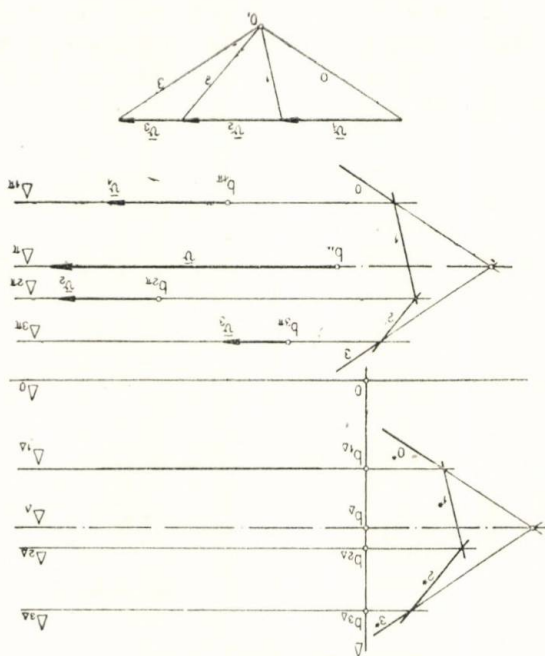


Fig. 4

La fig. 4 montre la réduction graphique effectuée par la construction de la ligne vectorielle des vecteurs $\bar{v}_{\mu}$ et par la construction de deux polygones funiculaires, l'un pour les vecteurs $\bar{v}_{\mu}$ appliqués dans les points $b_{\mu\Delta}$ situés sur l'axe Δ et l'autre pour les vecteurs $\bar{v}_{\mu}$ considérés dans le plan Π .

III. Centre d'un système de vecteurs parallèles

On démontre d'abord que les composantes plane-axielles du vecteur de position du centre C d'un système de vecteurs parallèles sont données par les relations :

$$(III.1) \quad \bar{r}_{C\Pi} = \frac{\sum_{\mu} \bar{r}_{\mu\Pi} v_{\mu}}{\sum_{\mu} v_{\mu}}$$

et

$$(III.2) \quad \bar{r}_{C\Delta} = \frac{\sum_{\mu} \bar{r}_{\mu\Delta} v_{\mu}}{\sum_{\mu} v_{\mu}}.$$

La première relation atteste que la projection sur le plan Π du centre des vecteurs v_{μ} coïncide avec le centre des vecteurs v_{μ} considérés dans le plan Π , le dernier centre pouvant être déterminé à l'aide de deux polygones funiculaires (fig. 5).

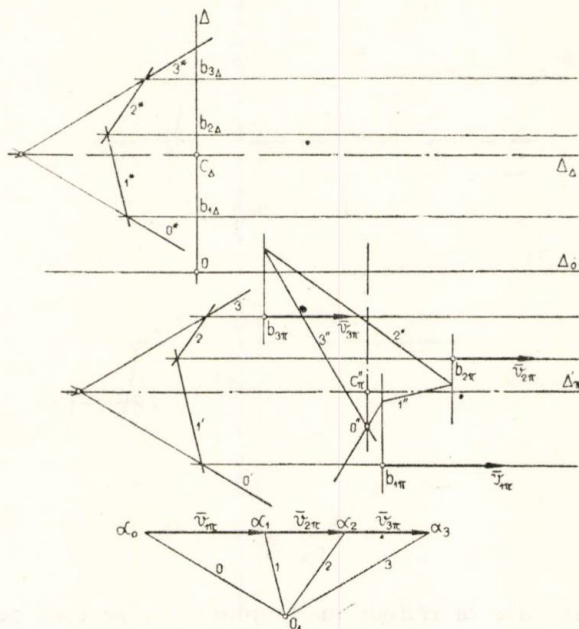


Fig. 5

La deuxième relation montre que la projection sur l'axe Δ du centre C coïncide avec le centre des vecteurs v_{μ} considérés appliqués dans les points $b_{\mu\Delta}$ sur l'axe Δ , ce dernier centre pouvant être déterminé aussi à l'aide d'un polygone funiculaire (fig. 5).

On peut vérifier que le procédé présenté est applicable aussi dans les conditions

$$\psi = 0 \quad \text{et} \quad \psi = \frac{\pi}{2} \quad (\text{rad}),$$

en rabattant tous les vecteurs, dans le dernier cas, sur des plans parallèles au plan II.

Reçue le 8 avril 1971

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O NOUĂ PREZENTARE GRAFICĂ A TEORIEI VECTORILOR ALUNECĂTORI

II. Reducerea sistemelor de vectori alunecători

(Rezumat)

Se prezintă o nouă metodă grafică, denumită de autori *metoda coordonatelor vectoriale plan-axiale*, de reducere a sistemelor de vectori alunecători. Folosindu-se, în cadrul acestei metode, numai procedee grafice corespunzătoare cazului plan — construcții de poligoane vectoriale și funiculare — se rezolvă expeditiv, simplu și suficient de precis, orice problemă de reducere a sistemelor de vectori alunecători oarecare și paraleli în spațiu.

ANISOTROPIC ALMOST CYLINDRICAL BEAMS:
THE BENDING BY A TRANSVERSE LOAD

BY

C. I. BORȘ

1. Introduction

We shall consider a beam limited by two planes $x_3 = 0$, $x_3 = h$ and a surface $\mathcal{F}$ given by

$$(1) \quad f[x_1(1 - kx_3), x_2(1 - kx_3)] = 0,$$

where k is a small parameter, the square and higher powers of which can be neglected.

Such beams were called *almost cylindrical beams* [1].

Many results are known concerning the beams of the shape (1). The homogeneous and the composite beams were taken into account in the isotropic case [1]...[3], etc., and also in the anisotropic case [4]...[6], etc.

In this Note for such beams we will study the problem of bending by a transverse load when the material of the beam is anisotropic with one plane of elastic symmetry perpendicular to the axis of x_3 . This problem was reduced at an Almansi-Michell's problem in the paper [5] but by complicated means.

Here, the solution of the problem of bending will be given in two steps:

(i) The problem will be reduced at an Almansi's problem by a very simple way.

(ii) The problem will be solved directly without to use of intermediate problems as Almansi-Michell's problem or Almansi's problem.

2. General Equations

We suppose that the surface $\mathcal{F}$ is free from tractions; also there are no body forces. We will denote by $\mathcal{O}$ the region occupied by the beam.

In this assumptions the stress components σ_{ij} must satisfy the equilibrium equations

$$(2) \quad \sigma_{ij,j} = 0 \quad \text{in } \mathcal{O}^1$$

and the boundary conditions

$$(3) \quad \sigma_{ij} n_j = 0 \quad \text{on } \mathfrak{F},$$

where n_i are the direction cosines of the exterior normal to surface $\mathfrak{F}$.

The traction applied on the end $x_3 = h$ are equivalent with a transverse load F_1 acting in the line of x_1 .

We will suppose that the material of the beam is anisotropic with one plane of elastic symmetry so that the Hooke's law may be written in the form

$$(4a) \quad \begin{cases} \sigma_{11} = A \gamma_{11} + H \gamma_{22} + G \gamma_{33} + Q \gamma_{12}, & \sigma_{22} = H \gamma_{11} + B \gamma_{22} + F \gamma_{33} + R \gamma_{12}, \\ \sigma_{33} = G \gamma_{11} + F \gamma_{22} + C \gamma_{33} + T \gamma_{12}, & \sigma_{12} = Q \gamma_{11} + R \gamma_{22} + T \gamma_{33} + D \gamma_{12}; \end{cases}$$

$$(4b) \quad \sigma_{23} = L \gamma_{23} + N \gamma_{31}, \quad \sigma_{31} = N \gamma_{23} + M \gamma_{31},$$

where $A, B, \dots, L, M$ are moduli of elasticity.

We will express the components of strain γ_{ij} in terms of stress components σ_{ij} by

$$(5a) \quad \begin{cases} \gamma_{11} = \frac{1}{E} (\nu_{11} \sigma_{11} + \nu_{12} \sigma_{22} + \nu_{13} \sigma_{12} - \nu_1 \sigma_{33}), \\ \gamma_{22} = \frac{1}{E} (\nu_{12} \sigma_{11} + \nu_{22} \sigma_{22} + \nu_{23} \sigma_{12} - \nu_2 \sigma_{33}), \\ \gamma_{12} = \frac{1}{E} (\nu_{13} \sigma_{11} + \nu_{23} \sigma_{22} + \nu_{33} \sigma_{12} - \nu_3 \sigma_{33}), \\ \gamma_{33} = \frac{1}{E} (-\nu_1 \sigma_{11} - \nu_2 \sigma_{22} - \nu_3 \sigma_{12} + \sigma_{33}); \end{cases}$$

$$(5b) \quad \gamma_{23} = \frac{1}{LM - N^2} (M \sigma_{23} - N \sigma_{31}), \quad \gamma_{31} = \frac{-1}{LM - N^2} (N \sigma_{23} - L \sigma_{31}),$$

where the coefficients of strain E, ν_{ij}, ν_i may be expressed in terms of moduli of elasticity [6].

¹⁾ The index j after comma indicates partial differentiation with respect to x_j . We use also the summation convention over the repeated indices.

The components γ_{ij} are given in terms of displacement components u_i by

$$\gamma_{ii} = u_{i,i} \text{ (not summed), } \gamma_{ij} = u_{i,j} + u_{j,i} \text{ (} i \neq j \text{)}$$

and they must satisfy the compatibility conditions of Saint-Venant

$$(6a) \quad \gamma_{11,22} + \gamma_{22,11} = \gamma_{12,12}, \dots$$

$$(6b) \quad (-\gamma_{23,1} + \gamma_{31,2} + \gamma_{12,3})_{,1} = \gamma_{11,23}, \dots$$

If we make the transformation

$$(7a) \quad \xi = x_1(1 - kx_3), \quad \eta = x_2(1 - kx_3), \quad \zeta = x_3;$$

$$(7b) \quad x_1 = \xi(1 + k\zeta), \quad x_2 = \eta(1 + k\zeta), \quad x_3 = \zeta,$$

then the surface (1) becomes

$$(8) \quad f(\xi, \eta) = 0,$$

which is a cylindrical surface $\mathcal{G}_1$ in the space ξ, η, ζ .

We will denote by S the domain of the cross-section of the cylindrical surface (8) and by Γ the boundary of S .

We can easily prove the following formulae

$$(9) \quad \begin{cases} \frac{\partial}{\partial x_1} = (1 - k\zeta) \frac{\partial}{\partial \xi}, & \frac{\partial}{\partial x_2} = (1 - k\zeta) \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial x_3} = \frac{\partial}{\partial \zeta} - k \left(\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \right) \end{cases}$$

and

$$(10) \quad n_1 = \cos \alpha, \quad n_2 = \cos \beta, \quad n_3 = -k(\xi \cos \alpha + \eta \cos \beta),$$

where $\cos \alpha, \cos \beta$ are the direction cosines of the exterior normal to the curve Γ .

We will take the axes of ξ, η, ζ such that the axis of ζ is central-line of the beam (8) and the axes of ξ and η are the principal axes of inertia of the end $\zeta = 0$. In this case we will have

$$(11) \quad \iint_S \xi \, d\sigma = 0, \quad \iint_S \eta \, d\sigma = 0, \quad \iint_S \xi \eta \, d\sigma = 0.$$

We will try to find the solution of mentioned problem supposing that the displacements u_i are of the form

$$(12) \quad \begin{cases} u_1 = -\tau\eta\zeta + a \left[\frac{1}{2} (h - \zeta)(\nu_1 \xi^2 - \nu_2 \eta^2) + \frac{1}{2} h \zeta^2 - \frac{1}{6} \zeta^3 \right] + k u_1^0, \\ u_2 = \tau \xi \zeta + a(h - \zeta) \left(\nu_2 \xi \eta + \frac{1}{2} \nu_3 \xi^2 \right) + k u_2^0, \\ u_3 = \tau \varphi(\xi, \eta) - a \left[\chi(\xi, \eta) + \left(h - \frac{1}{2} \zeta \right) \xi \zeta \right] + k u_3^0, \end{cases}$$

where u_i^0 are complementary unknown displacements, τ and a are constants which must be determined from the end conditions. Further, $\varphi(\xi, \eta)$ is the function of torsion of the beam (8) defined by

$$(13a) \quad \Delta \varphi = 0 \quad \text{in } S,$$

$$(13b) \quad \mathfrak{D} \varphi = (M\eta - N\xi) \cos \alpha - (L\xi - N\eta) \cos \beta \quad \text{on } \Gamma$$

and $\chi(\xi, \eta)$ is the flexion function of the same beam (8) defined by

$$(14a) \quad \Delta \chi + (M\nu_1 + L\nu_2 + N\nu_3 - E) \xi = 0 \quad \text{in } S.$$

$$(14b) \quad \begin{aligned} \mathfrak{D} \chi = & - \left[\frac{1}{2} M(\nu_1 \xi^2 - \nu_2 \eta^2) + N \left(\nu_2 \xi \eta + \frac{1}{2} \nu_3 \xi^2 \right) \right] \cos \alpha - \\ & - \left[\frac{1}{2} N(\nu_1 \xi^2 - \nu_2 \eta^2) + L \left(\nu_2 \xi \eta + \frac{1}{2} \nu_3 \xi^2 \right) \right] \cos \beta \quad \text{on } \Gamma. \end{aligned}$$

In the above formulae the operators Δ and $\mathfrak{D}$ are given by

$$(15) \quad \begin{cases} \Delta = M \frac{\partial^2}{\partial \xi^2} + 2N \frac{\partial}{\partial \xi \partial \eta} + L \frac{\partial^2}{\partial \eta^2}, \\ \mathfrak{D} = \cos \alpha \left(M \frac{\partial}{\partial \xi} + N \frac{\partial}{\partial \eta} \right) + \cos \beta \left(N \frac{\partial}{\partial \xi} + L \frac{\partial}{\partial \eta} \right). \end{cases}$$

The stresses corresponding to the displacements (12) are given by

$$(16) \quad \begin{cases} \sigma_{11} = -kG \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right) + k \tau_{11}, & \sigma_{22} = -kF \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right) + k \tau_{22}, \\ \sigma_{12} = -kT \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right) + k \tau_{12}, \end{cases}$$

$$\begin{aligned}
 (16) \quad \left\{ \begin{aligned}
 \sigma_{33} &= -a(1 - k\zeta)(h - \zeta)E\xi - kC\left(H_1 - \frac{1}{2}a\xi\zeta^2\right) + k\tau_{33}, \\
 \sigma_{23} &= L\left[\frac{\partial h_1}{\partial \eta} + \tau\xi - a\left(v_2\xi\eta + \frac{1}{2}v_3\xi^2\right)\right] + N\left[\frac{\partial h_1}{\partial \xi} - \tau\eta - \frac{1}{2}a(v_1\xi^2 - \right. \\
 &\quad \left. - v_2\eta^2)\right] - k\left\{L\left[\left(\frac{\partial h_1}{\partial \eta} + \tau\xi\right)\zeta + a(h - \zeta)(2v_2\xi\eta + v_3\xi^2)\right] + \right. \\
 &\quad \left. + N\left[\left(\frac{\partial h_1}{\partial \xi} - \tau\eta - a\left(h - \frac{1}{2}\zeta\right)\right)\zeta + a(h - \zeta)(v_1\xi^2 - v_2\eta^2)\right]\right\} + k\tau_{23}, \\
 \sigma_{31} &= N\left[\frac{\partial h_1}{\partial \eta} + \tau\xi - a\left(v_2\xi\eta + \frac{1}{2}v_3\xi^2\right)\right] + M\left[\frac{\partial h_1}{\partial \xi} - \tau\eta - \frac{1}{2}a(v_1\xi^2 - \right. \\
 &\quad \left. - v_2\eta^2)\right] - k\left\{N\left[\left(\frac{\partial h_1}{\partial \eta} + \tau\xi\right)\zeta + a(h - \zeta)(2v_2\xi\eta + v_3\xi^2)\right] + \right. \\
 &\quad \left. + M\left[\left(\frac{\partial h_1}{\partial \xi} - \tau\eta - a\left(h - \frac{1}{2}\zeta\right)\right)\zeta + a(h - \zeta)(v_1\xi^2 - v_2\eta^2)\right]\right\} + k\tau_{31},
 \end{aligned} \right.
 \end{aligned}$$

where τ_{ij} are the components of stress corresponding to the additional displacements u_i^0 and

$$(17) \quad h_1(\xi, \eta) = \tau\varphi(\xi, \eta) - a\chi(\xi, \eta), \quad H_1(\xi, \eta) = \xi \frac{\partial h_1}{\partial \xi} + \eta \frac{\partial h_1}{\partial \eta}.$$

A substitution of stresses (16) into equations (2) shows that the components τ_{ij} must satisfy the equations

$$\begin{aligned}
 (18) \quad \left\{ \begin{aligned}
 &\frac{\partial \tau_{11}}{\partial \xi} + \frac{\partial \tau_{12}}{\partial \eta} + \frac{\partial \tau_{31}}{\partial \zeta} - (G + M)\left(\frac{\partial H_1}{\partial \xi} - \frac{1}{2}a\xi^2\right) - (T + N)\frac{\partial H_1}{\partial \eta} - \\
 &- 2N[\tau\xi - a(2v_2\xi\eta + v_3\xi^2)] + 2M\{\tau\eta + a[(v_1\xi^2 - v_2\eta^2) + (h - \zeta)\zeta]\} = 0, \\
 &\frac{\partial \tau_{12}}{\partial \xi} + \frac{\partial \tau_{22}}{\partial \eta} + \frac{\partial \tau_{23}}{\partial \zeta} - (T + N)\left(\frac{\partial H_1}{\partial \xi} - \frac{1}{2}a\xi^2\right) - (F + L)\frac{\partial H_1}{\partial \eta} - \\
 &- 2L[\tau\xi - a(2v_2\xi\eta + v_3\xi^2)] + 2N\{\tau\eta + a[(v_1\xi^2 - v_2\eta^2) + (h - \zeta)\zeta]\} = 0, \\
 &\frac{\partial \tau_{31}}{\partial \xi} + \frac{\partial \tau_{23}}{\partial \eta} + \frac{\partial \tau_{33}}{\partial \zeta} - a\zeta[(2h - \zeta)(Mv_1 + Lv_2 + Nv_3 - E) + C\zeta] = 0.
 \end{aligned} \right.
 \end{aligned}$$

3. Reduction to Almansi's Problem

Let us take into account the solution of the equations (18) given by

$$(19) \left\{ \begin{aligned} \tau_{11}^* &= G \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right), & \tau_{22}^* &= F \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right), \\ \tau_{12}^* &= T \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right), & \tau_{33}^* &= C \left(H_1 - \frac{1}{2} a \xi \zeta^2 \right) - 2aE(h - \zeta) \xi \zeta, \\ \tau_{23}^* &= L \left[\left(\frac{\partial H_1}{\partial \eta} + 2\tau \xi \right) \zeta + a(h - 2\zeta)(2\nu_2 \xi \eta + \nu_3 \zeta^2) \right] + \\ &+ N \left[\left(\frac{\partial H_1}{\partial \xi} - 2\tau \eta \right) \zeta + a(h - 2\zeta)(\nu_1 \xi^2 - \nu_2 \eta^2) - a \left(h - \frac{1}{2} \zeta \right) \zeta^2 \right], \\ \tau_{31}^* &= N \left[\left(\frac{\partial H_1}{\partial \eta} + 2\tau \xi \right) \zeta + a(h - 2\zeta)(2\nu_2 \xi \eta + \nu_3 \zeta^2) \right] + \\ &+ M \left[\left(\frac{\partial H_1}{\partial \xi} - 2\tau \eta \right) \zeta + a(h - 2\zeta)(\nu_1 \xi^2 - \nu_2 \eta^2) - a \left(h - \frac{1}{2} \zeta \right) \zeta^2 \right]. \end{aligned} \right.$$

It is easy to see that the corresponding strains γ_{ij}^* satisfy the compatibility conditions (6).

Let now put down

$$(20) \quad \tau_{ij} = \tau_{ij}^* + \tilde{\tau}_{ij}.$$

From the equations (18) we will get that the new components of stress $\tilde{\tau}_{ij}$ must verify the equations

$$(21) \quad \tilde{\tau}_{ij,j} = 0^2).$$

If we take into account (16), (19), (20) and (13 b), (14 b) we will find that the new components of stress τ_{ij} satisfy the following boundary conditions

$$(22) \left\{ \begin{aligned} \tilde{\tau}_{11} \cos \alpha + \tilde{\tau}_{12} \cos \beta &= \left\{ M \left[\frac{\partial h_1}{\partial \xi} - \tau \eta - \frac{1}{2} a (\nu_1 \xi^2 - \nu_2 \eta^2) \right] + \right. \\ &\quad \left. + N \left[\frac{\partial h_1}{\partial \eta} + \tau \xi - a \left(\nu_2 \xi \eta + \frac{1}{2} \nu_3 \zeta^2 \right) \right] \right\} (\xi \cos \alpha + \eta \cos \beta), \\ \tilde{\tau}_{12} \cos \alpha + \tilde{\tau}_{22} \cos \beta &= \left\{ N \left[\frac{\partial h_1}{\partial \xi} - \tau \eta - \frac{1}{2} a (\nu_1 \xi^2 - \nu_2 \eta^2) \right] + \right. \\ &\quad \left. + L \left[\frac{\partial h_1}{\partial \eta} + \tau \xi - a \left(\nu_2 \xi \eta + \frac{1}{2} \nu_3 \zeta^2 \right) \right] \right\} (\xi \cos \alpha + \eta \cos \beta), \\ \tilde{\tau}_{31} \cos \alpha + \tilde{\tau}_{23} \cos \beta &= - (2\tau \mathcal{D}_\varphi - 3a \mathcal{D}_\chi - \mathcal{D}H_1) \zeta - \\ &\quad - aE\xi(h - \zeta) (\xi \cos \alpha + \eta \cos \beta) \quad \text{on } \mathfrak{F}_1, \end{aligned} \right.$$

²⁾ Here the indices 1, 2, 3 after comma indicate partial differentiation with respect to ξ, η, ζ .

where the operator $\mathfrak{D}$ is defined by (5).

The equations (21) and (22) define an Almansi's problem and we know how to solve it [7].

4. Direct Solution

Now, let us give a direct solution. In this purpose let us consider the following representation of the stresses τ_{ij} :

$$(23) \quad \begin{cases} \tau_{11} = \frac{\partial^2 \Phi}{\partial \eta^2} - M(\omega_1 + \theta_1) - N \int \frac{\partial \theta_2}{\partial \eta} d\xi + \tau_{11}^*, \\ \tau_{22} = \frac{\partial^2 \Phi}{\partial \xi^2} - L(\omega_1 + \theta_2) - N \int \frac{\partial \theta_1}{\partial \xi} d\eta + \tau_{22}^*, \\ \tau_{12} = -\frac{\partial^2 \Phi}{\partial \xi \partial \eta} - N\omega_1 + \tau_{12}^*, \\ \tau_{33} = \Omega + \frac{1}{2} E(A_1 \xi + A_2 \eta) \zeta^2 + \tau_{33}^*, \\ \tau_{23} = L \frac{\partial}{\partial \eta} [(\omega_1 + \theta_2) \zeta + \omega_0] + N \frac{\partial}{\partial \xi} [(\omega_1 + \theta_1) \zeta + \omega_0] + \tau_{23}^*, \\ \tau_{31} = N \frac{\partial}{\partial \eta} [(\omega_1 + \theta_2) \zeta + \omega_0] + M \frac{\partial}{\partial \xi} [(\omega_1 + \theta_1) \zeta + \omega_0] + \tau_{31}^*, \end{cases}$$

where Φ , ω_1 , ω_0 are unknown functions which will be defined below,

$$\begin{aligned} \Omega = E \left\{ \omega_1 + \frac{1}{2} \left[\theta_1 + \theta_2 + \left(\frac{1}{3} \nu_1 \xi + \frac{1}{2} \nu_3 \eta \right) A_1 \xi^2 + \left(\frac{1}{3} \nu_2 \eta + \frac{1}{2} \nu_3 \xi \right) A_2 \eta^2 \right] \right\} + \\ + \nu_1 \left[\frac{\partial^2 \Phi}{\partial \eta^2} - M(\omega_1 + \theta_1) - N \int \frac{\partial \theta_2}{\partial \eta} d\xi \right] + \nu_2 \left[\frac{\partial^2 \Phi}{\partial \xi^2} - L(\omega_1 + \theta_2) - N \int \frac{\partial \theta_1}{\partial \xi} d\eta \right] + \\ + \nu_3 \left[-\frac{\partial^2 \Phi}{\partial \xi \partial \eta} - N\omega_1 \right], \end{aligned}$$

$$\theta_1 = b_1 \xi \eta^2 - c_1 \xi \eta, \quad \theta_2 = a_1 \xi^2 \eta + c_1 \xi \eta,$$

A_i , a_i , b_i , c_i being some constants which will be chosen in such a way to satisfy the compatibility conditions and to ensure the existence of the functions ω_0 , ω_1 and Φ .

The stresses (23) will satisfy the equations (18) if the functions ω_0 and ω_1 verify the equations

$$(24a) \quad \Delta \omega_0 = 0 \quad \text{in } S$$

and

$$(25 \text{ a}) \quad \Delta \omega_1 + (EA_1 + 2Na_1)\xi + (EA_2 + 2Nb_1)\eta = 0 \quad \text{in } S.$$

From the third equation (3) we find for the functions ω_0 and ω_1 the following boundary conditions

$$(24 \text{ b}) \quad \mathfrak{D} \omega_0 = -ahE\xi(\xi \cos \alpha + \eta \cos \beta) \quad \text{on } \Gamma$$

and

$$(25 \text{ b}) \quad \begin{aligned} \mathfrak{D} \omega_1 = & -[N(a_1\xi^2 + c_1\xi) + M(b_1\eta^2 - c_1\eta)] \cos \alpha - \\ & -[L(a_1\xi^2 + c_1\xi) + N(b_1\eta^2 - c_1\eta)] \cos \beta + \\ & + 2\tau \mathfrak{D} \varphi - 3a \mathfrak{D} \chi - \mathfrak{D} H_1 + aE\xi(\xi \cos \alpha + \eta \cos \beta) \quad \text{on } \Gamma. \end{aligned}$$

All conditions of compatibility will be satisfied if

$$a_1 = \nu_1 A_2 - \frac{1}{2} \nu_3 A_1, \quad b_1 = \nu_2 A_1 - \frac{1}{2} \nu_3 A_2$$

and the function Φ verifies the equation

$$(26 \text{ a}) \quad \begin{aligned} & \beta_{22} \frac{\partial^4 \Phi}{\partial \xi^4} - 2\beta_{23} \frac{\partial^4 \Phi}{\partial \xi^3 \partial \eta} + (2\beta_{12} + \beta_{33}) \frac{\partial^4 \Phi}{\partial \xi^2 \partial \eta^2} - 2\beta_{13} \frac{\partial^4 \Phi}{\partial \xi \partial \eta^3} + \beta_{11} \frac{\partial^4 \Phi}{\partial \eta^4} = \\ & = (M\beta_{12} + L\beta_{22} + N\beta_{23} + \nu_2) \frac{\partial^2 \omega_1}{\partial \xi^2} - (M\beta_{13} + L\beta_{23} + N\beta_{33} + \nu_3) \frac{\partial^2 \omega_1}{\partial \xi \partial \eta} + \\ & + (M\beta_{11} + L\beta_{12} + N\beta_{13} + \nu_1) \frac{\partial^2 \omega_1}{\partial \eta^2} + 2[(M\beta_{11} + \nu_1)b_1\xi + L(\beta_{22} + \nu_2)a_1\eta] + \\ & + (N\beta_{12} - L\beta_{23})(2a_1\xi + c_1) + (N\beta_{13} - M\beta_{13})(2b_1\eta - c_1) \quad \text{in } S, \end{aligned}$$

where $\beta_{ij} = (\nu_{ij} - \nu_i \nu_j)/E$, ($i, j = 1, 2, 3$).

The boundary conditions of the function Φ , after (3), (16) and (23), are given by

$$(26 \text{ b}) \quad \left\{ \begin{aligned} & \frac{\partial^2 \Phi}{\partial \eta^2} \cos \alpha - \frac{\partial^2 \Phi}{\partial \xi \partial \eta} \cos \beta = (M \cos \alpha + N \cos \beta) \omega_1 + \\ & + \left[(Mb_1\eta - c_1)\xi\eta + N \left(\frac{1}{3} a_1\xi + \frac{1}{2} c_1 \right) \xi^2 \right] \cos \alpha + \\ & + \left\{ M \left[\frac{\partial h_1}{\partial \xi} - \tau\eta - \frac{1}{2} a(\nu_1\xi^2 - \nu_2\eta^2) \right] + \right. \\ & \left. + N \left[\frac{\partial h_1}{\partial \eta} + \tau\xi - a \left(\nu_2\eta + \frac{1}{2} \nu_3\xi \right) \xi \right] \right\} (\xi \cos \alpha + \eta \cos \beta), \end{aligned} \right.$$

$$\left[\begin{aligned} & -\frac{\partial^2 \Phi}{\partial \xi \partial \eta} \cos \alpha + \frac{\partial^2 \Phi}{\partial \xi^2} \cos \beta = (N \cos \alpha + L \cos \beta) \omega_1 + \\ & + \left[N \left(\frac{1}{3} b_1 \eta - \frac{1}{2} c_1 \right) \eta^2 + L(a_1 \xi + c_1) \xi \eta \right] \cos \beta + \\ & + \left\{ N \left[\frac{\partial h_1}{\partial \xi} - \tau \eta - \frac{1}{2} a(v_1 \xi^2 - v_2 \eta^2) \right] + \right. \\ & \left. + L \left[\frac{\partial h_1}{\partial \eta} + \tau \xi - a \left(v_2 \eta + \frac{1}{2} v_3 \xi \right) \xi \right] \right\} (\xi \cos \alpha + \eta \cos \beta) \text{ on } \Gamma. \end{aligned} \right.$$

From (26 a) and (26 b) it is obvious that the finding of the function Φ is similar with the finding of the Airy's function for the plane problems of anisotropic bodies with one plane of elastic symmetry.

From the conditions of existence of function Φ , using a similar way as in [8], we can determine the constants A_1 , A_2 , c_1 before to know the function ω_1 .

In this purpose we may use the following formulae

$$\begin{aligned} \int_{\Gamma} \omega_1 (M \cos \alpha + N \cos \beta) ds &= \int_{\Gamma} \xi \mathfrak{D} \omega_1 ds - \iint_S \xi \Delta \omega_1 d\sigma, \\ \int_{\Gamma} \omega_1 (N \cos \alpha + L \cos \beta) ds &= \int_{\Gamma} \eta \mathfrak{D} \omega_1 ds - \iint_S \eta \Delta \omega_1 d\sigma, \\ \int_{\Gamma} \omega_1 \mathfrak{D} \varphi ds &= \int_{\Gamma} \varphi \mathfrak{D} \omega_1 ds - \iint_S \varphi \Delta \omega_1 d\sigma. \end{aligned}$$

The solution given here will satisfy all equations and boundary conditions excepting the end conditions.

Therefore, it still remains to correct the end conditions by superposition of solution of some adequate problems for the cylindrical beam (8) but we can solve all these additional problems [6].

5. Some Remarks

1. Making $a = 0$ we obtain the solution for the problem of torsion.
2. The first step can be easily generalized to the case when the surface $\mathfrak{F}$ is of the form

$$(24) \quad f[x_1(1 - k\theta), x_2(1 - k\theta)] = 0,$$

where $\theta = \theta(x_3)$ is a given function of x_3 .

3. The second step can be done in a similar way and in the case when $\theta = x_3^2$ in the equation (24).

The case $\theta = x_3^2$ seems to be important because it includes the surfaces $ax_1^2 + bx_2^2 + kx_3^2 = c$, where a, b, c are constants.

Received January 6, 1971

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BARE ANIZOTROPE APROAPE CILINDRICE: ÎNCOVOIEREA DE CĂTRE O FORȚĂ TRANSVERSALĂ

(Rezumat)

Se consideră o bară aproape cilindrică mărginită de două plane $x_3 = 0$, $x_3 = h$ și de suprafața (1). Pentru o astfel de bară se studiază problema încovoierii de către o forță transversală.

În prima parte a lucrării se stabilesc ecuațiile generale ale problemei. În partea a doua problema studiată este redusă, pe o cale foarte simplă, la o problemă Almansi pentru bara cilindrică (8) pe care știm să o rezolvăm [7]. În sfârșit, în partea a treia se dă o rezolvare directă a problemei.

TRANSFORMATIONS POLYTROPIQUES HYPERSONIQUES D'UN FLUIDE IDÉAL

PAR

C. L. ACKER

Dans les travaux [1], [2] nous avons présenté le système différentiel extérieur des transformations générales polytropiques, qui a été obtenu en rapportant le mouvement du fluide idéal à un repère local triorthogonal $T(O; I_1, I_2, I_3)$ ayant son origine O à un point quelconque donné du domaine (D) occupé par le fluide, son vecteur unité I_3 étant égal à $\bar{v}/v$ ($\bar{v}$ est le vecteur vitesse) et I_2 étant, par exemple, normal au cône local des vitesses.

Nous allons utiliser ce système différentiel extérieur, en conservant même les notations des travaux cités.

Lorsqu'il s'agit du régime supersonique, ce système différentiel extérieur sera complété par l'équation

$$(s) \quad v = (1, \dots + F^2) \sqrt{w \rho^{w-1}}.$$

Pour faire l'étude des transformations polytropiques hypersoniques, nous avons attaché au système différentiel extérieur signalé l'équation

$$(h) \quad v = (K + F^2) \sqrt{w \rho^{w-1}},$$

où la valeur constante de K est une des plus faibles du nombre local de Mach, correspondant au régime hypersonique.

En examinant la classe générale des transformations polytropiques hypersoniques, nous avons considéré aussi les classes spéciales des transformations polytropiques hypersoniques soumises aux conditions suivantes géométriques ou mécaniques:

- 1) $v_t = 0$ (la condition par laquelle la grandeur du vecteur vitesse sera constante à un point quelconque du domaine occupé par le fluide; v sera donc stationnaire);

- 2) $v_t + vv_3 = 0$ (la condition par laquelle la dérivé substantielle de la grandeur du vecteur vitesse sera nulle);
- 3) $v_3 = 0$ (la condition par laquelle la grandeur du vecteur vitesse sera constante le long d'une ligne quelconque du courant);
- 4) $\text{grad } v = 0$ (la condition par laquelle, à un moment quelconque donné, la grandeur du vecteur vitesse conserve la même valeur à tous les point du domaine D occupé par le fluide);
- 5) $v = \text{const.}$;
- 6) $\rho_t + v\rho_3 = 0$ (la condition par laquelle la dérivé substantielle de la densité sera nulle);
- 7) $\rho_3 = 0$ (la condition par laquelle la densité sera constante le long d'une ligne quelconque du courant);
- 8) $\text{grad } \rho = 0$ (la condition par laquelle, à un moment quelconque donné, la densité conserve la même valeur à tous les points du domaine D occupé par le fluide);
- 9) $\rho = \text{const.}$

En attachant au système différentiel extérieur indiqué dans [1] (ou dans [2]) l'équation (h) aussi que les équations qui caractérisent chaque classe spéciale considérée, on a constitué successivement les systèmes différentiels extérieurs correspondants aux 14 classes de transformations examinées et on a cherché tout d'abord l'involution de ces systèmes différentiels extérieurs, pour faire un sondage sur l'existence et l'arbitraire des classes de transformations polytropiques hypersoniques considérées.

L'involution des systèmes différentiels extérieurs qu'on a utilisé, fut établie en construisant seulement les chaînes suivantes des éléments intégraux :

- a) $e_1(\omega^2 = \omega^3 = dt = 0, dv = v'_1 \omega^1, \dots),$
 $e_2(\omega^3 = dt = 0, dv = v'_1 \omega^1 + v'_2 \omega^2, \dots),$
 $e_3(dt = 0, dv = v'_1 \omega^1 + v'_2 \omega^2 + v'_3 \omega^3, \dots),$
 $e_4(dv = v'_1 \omega^1 + v'_2 \omega^2 + v'_3 \omega^3 + v'_t dt, \dots),$
- b) $e_1(\omega^1 = \omega^2 = \omega^3 = 0, dv = v'_t dt, \dots),$
 $e_2(\omega^2 = \omega^3 = 0, dv = v'_t dt + v'_1 \omega^1, \dots),$
 $e_3(\omega^3 = 0, dv = v'_t dt + v'_1 \omega^1 + v'_2 \omega^2, \dots),$
 $e_4(dv = v'_t dt + v'_1 \omega^1 + v'_2 \omega^2 + v'_3 \omega^3, \dots),$
- c) $e_1(\omega^2 = \omega^3 = \omega^1 = 0, dv = v'_t dt, \dots),$
 $e_2(\omega^3 = \omega^1 = 0, dv = v'_t dt + v'_2 \omega^2, \dots),$
 $e_3(\omega^1 = 0, dv = v'_t dt + v'_2 \omega^2 + v'_3 \omega^3, \dots),$
 $e_4(dv = v'_t dt + v'_2 \omega^2 + v'_3 \omega^3 + v'_1 \omega^1, \dots).$

Les résultats concernant l'existence et l'arbitraire des classes de transformations polytropiques hypersoniques, qu'on a établie, seront présentés ci-dessous d'une manière brève.

Par exemple

$$\text{PH1a}; 9, 5, 5, 5; 2f(4), \Phi_1, r_t/fmq; 5f(3), v_3, \rho_3, p_3, q_3, r_3/\Phi_1, r_t \text{ pr.}$$

pe qui signifie: on a établie l'existence de la classe des transformations *colytropiques hypersoniques soumises à la condition (1)*, en construisant la chaîne des éléments intégraux a); on a trouvé les caractères réduits $s_0 = 9$, $s_1 = s_2 = s_3 = 5$; cette classe dépend de 2 fonctions arbitraires de 4 arguments, ces fonctions étant Φ_1 et r_t lorsque les forces massiques restent quelconques; la même classe dépend de 5 fonctions arbitraires de 3 arguments ($v_3, \rho_3, p_3, q_3, r_3$) si les fonctions Φ_1 et r_t sont prescrites.

Un autre exemple

$$\text{PH4, 8, c}; 7, 5, 3, 3; 1(f(4)/fmq,$$

ce qui signifie: on a établie l'existence de la classe des transformations *polytropiques hypersoniques soumises aux conditions 4 et 8*, en construisant la chaîne des éléments intégraux c); on a trouvé les caractères réduits $s_0 = 7$, $s_1 = 5$, $s_2 = s_3 = 3$; cette classe dépend de 1 fonction arbitraire de 4 arguments si les forces massiques restent quelconques.

Voici ces résultats.

Cl. 1) PHa; 8, 5, 5, 5; 3f(4), $\Phi_3, \Phi_1, r_t/fmq; 5f(3), v_3, \rho_3, p_3, q_3, r_3/\Phi_3, \Phi_1, r_t \text{ pr}$

Cl. 2) PH1a; 9, 5, 5, 5; 2f(4), $\Phi_1, r_t/fmq; 5f(3), v_3, \rho_3, p_3, q_3, r_3/\Phi_1, r_t \text{ pr}$

Cl. 3) PH2a; 9, 5, 5, 5; 2f(4), $\Phi_1, r_t/fmq; 5f(3), v_3, \rho_3, p_3, q_3, r_3/\Phi_1, r_t \text{ pr}$

Cl. 4) PH3b; 8, 5, 5, 5; 2f(4)/fmq

Cl. 5) PH4b; 7, 5, 4, 4; 2f(4)/fmq

Cl. 6) PH5a; 7, 4, 4, 4; 2f(4), $\Phi_1, r_t/fmq; 4f(3), \rho_3, p_3, q_3, r_3/\Phi_1, r_t \text{ pr}$

Cl. 7) PH6b) 8, 5, 5, 5; 2f(4)/fmq.

Cl. 8) PH7b; 8, 5, 5, 5; 2f(4) fmq

Cl. 9) PH8b; 7, 5, 4, 4; 2f(4) fmq

Cl. 10) PH9b; 6, 4, 4, 4; 2f(4)/fmq

Cl. 11) PH 4, 8, c; 7, 5, 3, 3; 1f(4)/fmq

Cl. 12) PH 4, 9, c; 6, 4, 3, 3; 1f(4)/fmq

Cl. 13) PH 5, 8, c; 6, 4, 3, 3; 1f(4)/fmq

Cl. 14) PH 5, 9, c; 5, 3, 3, 3; 1f(4)/fmq

Observations. 1. Les prolongements du 2-ème ordre ne sont pas nécessaires pour mettre en évidence l'involution des systèmes différentiels extérieurs utilisés.

2. Les transformations polytropiques sont caractérisées dans la classe des transformations barotropes par l'expression suivante du carré c^2 de la vitesse du son :

$$c^2 = \frac{wb}{\rho},$$

où w est le rapport des chaleurs spécifiques C_p/C_v et b est la pression. En effet, la relation précédente entraîne l'équation $db/d\rho = wb/\rho$, dont la solution générale est $b = C\rho^w$.

3. Les transformations polytropiques sont caractérisées aussi dans la classe des transformations barotropes par l'expression suivante de c^2

$$c^2 = wb \frac{d}{dt} (V_s) \Big/ \operatorname{div} \bar{v},$$

où V_s est le volume spécifique.

4. Pour les classes de transformations polytropiques soumises du moins à la condition (3), la valeur de c^2 est donnée par la relation

$$c^2 = -wb \frac{d}{dt} (V_s) \Big/ \left[v R_m \left(\frac{\bar{v}}{v} \right) \right],$$

$\left(R_m \left(\frac{\bar{v}}{v} \right) \right)$ est le rayon de la courbure moyenne du champ $\frac{\bar{v}}{v}$, qui caractérise la Cl. 4 dans la Cl. 1.

5. Les transformations polytropiques sont caractérisées dans la classe des transformations barotropes, aussi par chacune des deux expressions suivantes du nombre local de Mach :

$$M^2 = \frac{v^2 \rho}{wb}, \quad M^2 = v^2 \operatorname{div} \bar{v} \Big/ \left[wb \frac{d}{dt} (V_s) \right].$$

6. Pour les classes de transformations polytropiques soumises du moins à la condition (3), la valeur du nombre local de Mach est donnée par la relation

$$M^2 = -v^3 R_m \left(\frac{\bar{v}}{v} \right) \Big/ \left[wb \frac{d}{dt} (V_s) \right],$$

cette relation caractérisant aussi la Cl. 4 dans la Cl. 1.

7. Les transformations hypersoniques sont caractérisées dans la Cl. 1 par chacune des deux inéquations suivantes :

$$v^2 \rho \geq Kwb, \quad \operatorname{div} \bar{v} \Big/ \frac{d}{dt} (V_s) \geq \frac{Kwb}{v^2},$$

tandis que dans les classes de transformations polytropiques soumises du moins à la condition (3), les transformations hypersoniques sont caractérisées par l'inéquation

$$-R_m \left(\frac{\bar{v}}{v} \right) \bigg/ \frac{d}{dt} (V_s) \geq \frac{Kwb}{v^3}.$$

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TRANSFORMĂRI POLITROPICE HIPERSONICE ALE UNUI FLUID IDEAL

(Rezumat)

Se stabilesc rezultate privind existența și arbitraritatea transformărilor politropice hipersonice, prin utilizarea teoriei sistemelor diferențiale exterioare (E. Cartan).

Sînt semnalate de asemenea unele proprietăți caracteristice ale acestor transformări.

LA CONCEPTION QUANTIQUE NONRELATIVISTE DU CHAMP ÉLECTROMAGNÉTIQUE ET DES MICROPARTICULES CHARGÉES (II)

PAR

LIVIU SOFONEA

1. Introduction

Dans la première partie de cet étude [1] nous avons analysé les propriétés les plus générales du champ électromagnétique et de son interaction avec les systèmes des microparticules, tout en considérant le caractère quantique du mouvement.

Nous continuons ici l'étude de l'interaction du champ électromagnétique avec les systèmes quantiques, en nous limitant seulement aux effets non relativistes.

2. Le caractère quantique de la radiation

Pour comprendre la valabilité de l'image nonquantifiée du champ électromagnétique, les résultats du calcul de la radiation multipolaire [2], [3], [4], [5] (effectués surtout pour les transitions γ) sont soumis à une interprétation particulière. On considère maintenant que le photon qui est lié d'une manière inéluctable à chaque onde classique se trouve dans des états physiques bien déterminés. Le photon possède un moment cinétique et une parité bien déterminées [6]. Les états de moment cinétique sont décrites par une fonction d'onde vectorielle dont les propriétés sont établies sans quantifier pour la deuxième fois le champ [7], [6]. Dans cette théorie quantique initiale on prend simplement la moyenne des résultats classiques contenus dans l'équation de Maxwell pour le vacuum (l'énergie, l'impulsion, le moment cinétique)

$$(2.1) \quad \vec{\Psi}_{JlMs}(\vec{k}) = a(k) \sum_{\mu=-1}^1 \langle C_{J,M-\mu;s\mu}^{l,M-\mu;s\mu} \rangle Y_{l,M-\mu}(\vec{k}) \vec{x}_{\mu} \equiv a(k) \vec{Y}_{JlM}^{(1)},$$

La partie angulaire porte le nom de vecteur sphérique $\vec{Y}_{JlM}$; $\vec{x}_{s\mu} \equiv \vec{x}_{\mu}$ est la fonction de spin, étroitement liée au vecteur de polarisation $\vec{1}_{\mu} = \vec{x}_{\mu}$ qui peut prendre seulement deux valeurs; J désigne ici le moment cinétique total $\hbar J(J+1)$, l est maintenant le nombre quantique „orbital“ $l = J, J \pm 1$; $J \neq 0$, nombre qui n'est pas toujours „bon“; M nous montre la projection du moment total sur une axe quelconque $Oz(-J, \dots, 0, \dots, +J)$; m sont les projections du moment „orbital“ $(-l, \dots, 0, \dots, +l)$; μ sont des nombres quantiques de projection pour le spin $\mu = 0, \pm 1$; $\mu = M - m$ les nombres μ précisent l'état de polarisation d'une onde $\mu = \pm 1$. La valeur $\mu = 0$ ne peut décrire aucune état physique de polarisation, car cette état formel est rendu impossible par la condition de transversalité; $Y_{lM}^M(\theta\varphi)$ est la fonction sphérique (fonction propre de l'opérateur moment „orbital“; elle dépend des angles du verneur $\vec{k}$ (impulsion d'onde)); $C_{J,M-\mu;s\mu}^{l,M-\mu;s\mu}$ est une matrice à six éléments non nuls qui correspondent aux indices $\mu = -1, 0, 1$; $J = l \pm 1, l$; $S = 1$ est le spin du photon.

Les fonctions sphériques vectorielles ont la forme :

$$(2.2) \quad \vec{Y}_{JlM}(\theta\varphi) = \sum_{m=-l}^{m=+l} \sum_{\mu=-1}^{+s} \langle l, 1, m, \mu | JM \rangle Y_{lm}(\theta\varphi) \vec{x}_{\mu}$$

(les coefficients écrits à l'aide de la notation matricielle sont connus sous le nom de coefficients Klebsch-Gordon).

L'état d'un photon ayant le moment cinétique déterminé J, M présente un double degré de dégénérescence [6] qui se précise à l'aide d'un nouveau indice de parité λ qui prend deux valeurs. Pour J et M donnés sont possibles seulement deux états avec des parités différentes: état de type électrique avec $\lambda = 1$ et état de type magnétique avec $\lambda = 0$. Le nombre quantique l ne peut indiquer un état bien déterminé car $\vec{Y}_{JM}^{(+1)}$ est une combinaison linéaire de deux vecteurs sphériques avec des valeurs J différentes ($J \pm 1$). L'état $J = 0$ ne se réalise pas: il ne convient pas à la transversalité. Nous avons utilisé d'ailleurs cette condition dans [1, § 6]. Par rapport à la parité des champs les états sont de deux espèces [6]:

États de type électrique ($\lambda = 1$):

$$(2.3) \quad \vec{\mathcal{E}}_{E,J,M,+1} = [A(kr) \vec{Y}_{J,J+1,M} + B(kr) \vec{Y}_{J,J-1,M}] e^{-i\omega t}, \quad \Pi = (-1)^{J+1};$$

$$(2.4) \quad \vec{\mathcal{H}}_{E,J,M,+1} = C(kr) \vec{Y}_{J,J,M} e^{-i\omega t}, \quad \Pi = (-1)^J.$$

États de type magnétique ($\lambda = 0$):

$$(2.5) \quad \vec{\mathcal{E}}_{E, J, M, 0} = D(kr) \vec{Y}_{J, J, M} e^{-i\omega t}, \quad \Pi = (-1)^J;$$

$$(2.6) \quad \vec{\mathcal{H}}_{E, J, M, 0} = [F(kr) \vec{Y}_{J, J+1, M} + G(kr) \vec{Y}_{J, J-1, M}] e^{-i\omega t}, \quad \Pi = (-1)^{J+1},$$

(Π désigne la parité).

Les champs possèdent une variation angulaire spécifique parce que la parité des trois termes du développement du champ selon les fonctions sphériques vectorielles n'est pas la même. Pour la radiation électrique qui possède l'ordre de multipôle J, M le champ magnétique (exprimé par une

seule fonction vectorielle $\vec{Y}_{J, J, M}^{(\lambda=1)}$ a la parité $(-1)^J$ et par conséquent les

expressions dans lesquelles sont présentes les autres fonctions $\vec{Y}_{J\pm 1, M}$ (qui conditionnent le champ électrique) doivent avoir les coefficients nuls. Le champ magnétique sera :

$$(2.7) \quad \mathcal{H}(E_{J, M}; \vec{r}) = \frac{a(E_{J, M})}{kr} g_{el}(J, M; r) \vec{Y}_{J, M}(\theta, \varphi),$$

$g_{el}(r)$ est la fonction radiale, $\vec{Y}_{J, M}$ la fonction sphérique vectorielle qui envisage le multipôle. La fonction radiale g_{el} est donnée par l'équation d'onde

$\Delta \mathcal{H} = 0$ dans laquelle on doit faire la substitution (dans le résultat final on doit mettre $r \rightarrow \infty$). Le champ électrique est fourni par l'équation de Maxwell $\vec{\mathcal{E}} = -\frac{i}{k} \text{rot } \vec{\mathcal{H}}$. On obtient :

Radiation de type électrique ;

$$(2.8) \quad \vec{\mathcal{H}}(E_{J, M}; \vec{r}) = a(E_{J, M}) \frac{g_{el}(J, r)}{kr} \vec{Y}_{J, M}(\theta, \varphi),$$

$$(2.9) \quad \vec{\mathcal{E}}(E_{J, M}; \vec{r}) = a(E_{J, M}) \frac{i}{k^2} \text{rot} \left[\frac{g_{el}(J, r)}{r} \vec{Y}_{J, M}(\theta, \varphi) \right].$$

Radiation de type magnétique :

$$(2.10) \quad \vec{\mathcal{E}}(M_{J, M}; \vec{r}) = a(E_{J, M}) \frac{g_{mag}(J, r)}{kr} \vec{Y}_{J, M}(\theta, \varphi),$$

$$(2.11) \quad \vec{\mathcal{H}}(M_{J, M}; \vec{r}) = \frac{-i}{k^2} a(M_{J, M}) \text{rot} \left[\frac{g_{mag}(J, r)}{r} \vec{Y}_{J, M}(\theta, \varphi) \right].$$

Les propriétés de transversalité des champs sont désignées par la fonction $\vec{Y}_{J, M}$ (normale au vecteur de position dans chaque point de la

sphère de rayon r). La condition $\mathcal{J} \neq 0$ signifie l'absence d'un état photonique dépourvu d'un moment cinétique: chaque quanta transporte son moment cinétique bien déterminé par son multipôle.

Précisons maintenant les sources (courants, densités de charge et magnétisations $\vec{\mathcal{M}}$, qu'on suppose être soumises à une oscillation harmonique: $\vec{j}(\vec{r}, t) = \vec{j}(\vec{r})e^{-i\omega t} + \vec{j}^*(\vec{r})e^{+i\omega t}$ etc. Les champs $\vec{\mathcal{E}}$ et $\vec{\mathcal{H}}$ qui sont en corrélation doivent obéir à des équations d'onde non homogènes (renfermant les sources des champs)

$$(2.12) \quad \text{rot rot } \vec{\mathcal{H}}(\vec{r}) - k^2 \vec{\mathcal{H}}(\vec{r}) = \frac{4\pi}{c} (\text{rot } \vec{j} + ck^2 \vec{\mathcal{M}}),$$

$$(2.13) \quad \text{rot rot } \vec{\mathcal{E}}(\vec{r}) - k^2 \vec{\mathcal{E}}(\vec{r}) = \frac{4\pi ik}{c} (\vec{j} + c \text{rot } \vec{\mathcal{M}}).$$

Il faut remplacer (2.7) (pour les deux types de radiation) et alors prendre l'intégrale (dans la zone d'ondes) par rapport aux fonctions sphériques vectorielles (orthogonales). Si on compare les expressions obtenues avec les expressions asymptotiques valables dans le cas d'absence des sources on peut déterminer la forme des fonctions $a(E_{J,M})$. On cherche à trouver la solution de l'équation des ondes nonhomogènes par la méthode de la fonction de Green. On peut obtenir des expressions assez simples pour la région très lointaine (et dans l'approximation $kr \ll 1$) [8] ... [10]

$$(2.14) \quad a(E_{J,M}) \approx \frac{4\pi}{(2\mathcal{J}+1)!!} \frac{(\mathcal{J}+1)^{1/2}}{\mathcal{J}} k^{J+2} [\mathfrak{X}(E_{J,M}) + \mathfrak{X}'(E_{J,M})],$$

$$(2.15) \quad a(M_{J,M}) \approx \frac{4\pi}{(2\mathcal{J}+1)!!} \left(\frac{\mathcal{J}+1}{\mathcal{J}} \right)^{1/2} k^{J+2} [\mathfrak{X}(M_{J,M}) + \mathfrak{X}'(M_{J,M})].$$

Le symbole!! représente le produit des nombres impaires jusqu'à $2\mathcal{J}+1$; $\mathfrak{X}(E_{J,M})$ s'appelle le moment électrique multipolaire et $\mathfrak{X}(M_{J,M})$ le moment magnétique multipolaire. Les quantités $\mathfrak{X}'(E_{J,M})$ et $\mathfrak{X}'(M_{J,M})$ ont pris le même nom:

$$(2.16) \quad \mathfrak{X}(E_{J,M}) = \int r^J Y_{J,M}^*(\theta, \varphi) \rho(\vec{r}) d\vec{r},$$

$$(2.17) \quad \mathfrak{X}(M_{J,M}) = -\frac{1}{c(\mathcal{J}+1)} \int r^J Y_{J,M}^*(\theta, \varphi) \text{div}(\vec{r} \times \vec{j}) d\vec{r},$$

$$(2.18) \quad \mathfrak{X}'(E_{J,M}) = -\frac{ik}{\mathcal{J}+1} \int r^J Y_{J,M}^*(\theta, \varphi) \text{div}(\vec{r} \times \vec{\mathcal{M}}) d\vec{r},$$

$$(2.19) \quad \mathfrak{X}'(M_{J,M}) = -\int r^J Y_{J,M}^*(\theta, \varphi) \text{div } \vec{\mathcal{M}} d\vec{r}.$$

Pour $J = 1$, $M = 0$ on obtient les expressions connues pour les moments dipolaires (électrique et magnétique):

$$(2.20) \quad \mathfrak{K}(E_{1,0}) = \sqrt{\frac{3}{4\pi}} \int r \rho(r) (\vec{dr}),$$

$$(2.21) \quad \mathfrak{K}'(E_{1,0}) = \sqrt{\frac{3}{4\pi}} \frac{ik}{2} \int (\vec{r} \times \vec{\mathcal{M}})_z (\vec{dr}),$$

$$(2.22) \quad \mathfrak{K}(M_{1,0}) = \sqrt{\frac{3}{4\pi}} \frac{i}{2c} \int (xj_y - yj_x) (\vec{dr}),$$

$$(2.23) \quad \mathfrak{K}(M_{1,0}) = \sqrt{\frac{3}{4\pi}} \int \vec{\mathcal{M}}_z (\vec{dr}).$$

Les probabilités d'émission dans une seconde d'une radiation (d'un certain type) sont données par le rapport entre la valeur du vecteur rayonnant et l'énergie de la quanta. Le vecteur $\vec{S}$ s'obtient tout de suite car le produit $\vec{\mathcal{E}} \times \vec{\mathcal{H}}$ se construit à l'aide de (2.8)...(2.11) (pour une région lointaine).

Opérons maintenant une intégrale par rapport aux angles:

$$(2.24) \quad dW(E_{J,M}; \theta, \varphi) = \frac{c}{2\pi k^2} |a(E_{J,M})|^2 \vec{Y}_{J,M}^* \vec{Y}_{J,M} d\Omega,$$

$$(2.25) \quad dW(M_{J,M}; \theta, \varphi) = \frac{c}{2\pi k^2} |a(M_{J,M})|^2 \vec{Y}_{J,M}^* \vec{Y}_{J,M} d\Omega,$$

$$(2.26) \quad W(E_{J,M}) = \frac{c}{2\pi k^2} |a(E_{J,M})|^2,$$

$$(2.27) \quad W(M_{J,M}) = \frac{c}{2\pi k^2} |a(M_{J,M})|^2.$$

Les probabilités sont proportionnelles aux facteurs d'amplitude (conditionnées par les sources (7.14), (7.15), qui contiennent des termes quadratiques qu'on peut réunir dans des expressions distinctes nommées probabilités réduites [9], [8]. L'analyse de ces probabilités réduites nous montre que l'émission d'une radiation dipolaire est préférable par rapport à la radiation quadrupolaire. (De même pour d'autres détails, surtout pour les états faiblement excités et pour des grandes longueurs d'onde).

Les règles de sélection sont donnés par la loi quantique de composition du moment cinétique. La radiation transporte un moment à nombre quantique qui est toujours „bon“ et qui doit obéir à la restriction

$$(2.28) \quad |\mathcal{J}_i - \mathcal{J}_f| \leq \mathcal{J} \leq \mathcal{J}_i + \mathcal{J}_f, \quad M_i - M_f = M.$$

Parce que l'état $\mathcal{J} = 0$ est impossible les transitions $\mathcal{J}_i = \mathcal{J}_f = 0$ sont interdites.

Les conditions dans lesquelles se trouve le système sont reflétées dans les propriétés de parité de la fonction d'onde (parité qui se conserve pendant le mouvement). La probabilité de transition est proportionnelle à l'élément de matrice de la perturbation $\rho \sim \int \Psi_f^* \vec{j} \vec{\mathcal{A}} \Psi_i(\vec{r}) d\vec{r}$ expression dans laquelle le vecteur polaire $\vec{j}$ a la parité (-1) et $\vec{\mathcal{A}}$ la parité du champ électrique. Donc le produit scalaire $\vec{j} \vec{\mathcal{A}}$ a la parité du champ magnétique. La transition est interdite si la fonction à intégrer est impaire. Pour la radiation de type électrique la parité de $\vec{\mathcal{H}}$ est $\Pi = (-1)^J$, et pour un type magnétique $(-1)^{J+1}$. Donc la transition peut avoir lieu si la parité de la fonction d'onde ne change pas quand $\mathcal{J}$ est pair; ou pour le cas d'un fonction d'onde changeant la parité pour un $\mathcal{J}$ impair (nous nous avons fixée l'attention sur le cas d'émission d'une radiation de type électrique; pour une radiation magnétique la situation inverse).

L'analyse des multipôles faite à l'aide des données fournies par la spectroscopie du noyau [8], [11], [12], montre que les radiations électriques sont générées dans la majeure partie par les mouvements des charges des sources. Les radiations de type magnétique sont générées par les moments magnétiques (d'origine orbitale et de spin) et par conséquent, les transitions de type magnétique ($M, \mathcal{J}$) sont moins intenses que celles électriques ($E, \mathcal{J}$) (du même degré de multipolarité). Le rapport des probabilités est donné par le carré du rapport entre le moment magnétique et le moment électrique. Pour déterminer ce nombre il faut faire des hypothèses sur la structure du système (modèle du noyau). Les règles de sélection par rapport à la parité indiquent encore que le mélange entre les radiations d'un même degré de multipolarité $\mathcal{J}$ est impossible. La dépendance de la probabilité par rapport à $\mathcal{J}$ est donnée par une fonction qui décroît d'une manière très rapide, ce qui nous montre que si les états (excité et fondamental) possèdent des spins très différents, la chance d'une radiation (en supposant que l'excitation est faible) est très petite. De la sorte on peut entreprendre un étude de l'isomerie du noyau [11]...[13].

Pour le cas dipolaire $\int \Psi_m^* \vec{r} \Psi_m d\vec{r}$ sera nul quand $\Psi_m^* \Psi_m$ sera une fonction impaire (car $\vec{r}$ est une fonction impaire). Le produit $\Psi_m^* \Psi_m$ sera impair seulement quand la parité de la fonction qui décrit l'état initial est différente de celle de la fonction qui décrit l'état final: pour les transitions électriques dipolaires la parité de la fonction d'onde doit changer (c'est le cas de différentes couches du noyau). Pour la transition quadru-

polaire le tenseur moment électrique quadrupolaire contient le produit $\vec{j} \vec{r}'$ des deux vecteurs polaires $\vec{j}, \vec{A}$, qui est une fonction paire. L'élément de matrice sera différent de zéro si $\Psi_m^* \Psi_n$ est une fonction paire: la parité de la fonction d'onde ne doit pas changer dans les transitions quadrupolaires (c'est le cas d'une couche nucléaire dont les états sont décrits par des fonctions avec la même parité). Pour la radiation magnétique dipolaire, dans l'expression intervient le moment magnétique du courant (un vecteur axial), donc une fonction paire. Par la suite $\Psi_m^* \Psi_n$ doit être encore une fonction paire. Pour la radiation magnétique la parité de la fonction d'onde restera invariable. La situation est la même pour la radiation dipolaire magnétique car l'opérateur $\hat{\mu}_\sigma$ contient les matrices σ qui sont des nombres; (évidemment ils ne changent pas leur signe si l'on fait une inversion d'axes (quantité paire).

3. Conclusions

La description d'ensemble faite ici pour l'interaction du champ électromagnétique avec les systèmes quantiques se limite à des effets non-relativistes. Pour pouvoir comprendre les effets relativistes il faut faire appel à d'autres méthodes (nombre variable des microparticules, double quantification du champ, etc.) propre à l'électrodynamique [14] ... [16] quantique. Mais les idées directrices d'une telle analyse dépassent de loin le moment historique de la pensée que nous avons essayé de présenter ici.

Reçue le 14 novembre 1969

Institut Polytechnique, Braşov,
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CONCEPȚIA CUANTICĂ NERELATIVISTĂ ASUPRA CÎMPULUI ELECTROMAGNETIC ȘI MICROPARTICULELOR ÎNCĂRCATE (II)

(Rezumat)

Se continuă analiza începută în [1] privind proprietățile cele mai generale ale cîmpului electromagnetic și ale interacțiunii sale cu sistemele de microparticule. Se insistă asupra modului în care intervin în mod necesar conceptele clasice și cele cuantice (sau numai semicuantice) atît în privința mișcării microparticulei (care este considerată în mod consecvent cuantic) cît mai ales în privința cîmpului electromagnetic. Se stabilesc raporturile atît cu electrodinamica clasică cît și cu cea cuantică bazată pe ideea cuantificării cîmpului și a renormării parametrilor de structură.

SUR LA PROPAGATION DES ONDES MAGNÉTOACOUSTIQUES DANS UN MAGNÉTOPLASMA EN ÉQUILIBRE À DES NONHOMOGÉNÉITÉS LINÉAIRES

PAR

D. L. GOLDENBERG et R. V. DEUTSCH

1. L'étude des oscillations globales des systèmes magnétohydrodynamiques non homogènes sous la forme la plus générale est un problème extrêmement difficile. On peut démontrer que, seulement pour certains systèmes particuliers, la propagation des oscillations globales de type Alfvén ou magnétoacoustiques est possible (la condition nécessaire et suffisante pour la réalisation du découplage est donnée en [1]).

Même au cas où cette condition est réalisée la solution du problème de la propagation des perturbations magnétoacoustiques peut être trouvée seulement pour des structures ayant une certaine symétrie est dans certaines hypothèses restrictives.

Ainsi, lorsque la vitesse de phase est indépendante de l'orientation du vecteur d'onde par rapport à la direction du champ magnétique on peut appliquer l'approximation de l'optique géométrique et, en particulier, le principe de Fermat. En ce cas, les trajectoires des „rayons“ sont déterminées par une équation de type Hamilton-Jacobi.

En ce qui concerne les ondes magnétoacoustiques rapides qui se propagent dans des systèmes où la condition

$$(1.1) \quad U \gg c_0$$

est réalisé, on peut écrire pour la vitesse de phase l'expression approximative

$$(1.2) \quad v_{\Phi+} \approx U$$

(par U nous avons noté la vitesse Alfvén; c_0 est la vitesse du son).

Dans ces hypothèses, plusieurs ouvrages récemment publiés ([2] ... [4]) analysent le comportement des ondes magnétoacoustiques se propageant dans des champs magnétiques à symétrie axiale. Une conclusion commune pour

tous ces ouvrages consiste dans la profondeur limitée de pénétration des ondes dans les domaines à champ magnétique croissant.

On montre que dans ces domaines il y a des surfaces caractéristiques sur lesquelles les ondes magnétoacoustiques semblent subir une réflexion totale.

Serait-ce un résultat caractéristique pour les champs à symétrie axiale? Dans cet article nous allons prouver que la réponse à cette question est négative.

2. Nous allons considérer un plasma situé dans un champ magnétique légèrement non homogène et dont la seule composante B est une fonction d'une seule coordonnée spatiale (cf. fig. 1)

$$(2.1) \quad B_x = B = B_0(1 + \alpha z), \quad B_y = B_z = 0$$

(ici α est un paramètre petit qui assure la variation lente de B),

$$(2.2) \quad \alpha \ll 1.$$

En tenant compte de la relation

$$(2.3) \quad p + \frac{B^2}{2\mu_0} = p_0 + \frac{B_0^2}{2\mu_0}$$

et en utilisant l'équation d'état des gaz idéaux, on trouve que la densité du gas varie selon la loi

$$(2.4) \quad \rho = \rho_0(1 - \gamma z),$$

où γ dénote l'expression

$$(2.5) \quad \gamma = \frac{\alpha U_0^2 M}{RT},$$

et U_0 est la vitesse Alfvén dans le cas limite $\alpha = 0$.

Conformément aux relations (2.1) et (2.4), (2.5) nous obtenons pour la vitesse Alfvén l'expression

$$(2.6) \quad U = U_0(1 + \lambda z) + \text{des termes d'ordre supérieur en } \lambda,$$

où par λ nous avons noté

$$(2.7) \quad \lambda = \alpha + \frac{\gamma}{2} = \alpha \left(1 + \frac{U_0^2 M}{RT} \right) \ll 1.$$

De suite, nous allons considérer seulement des perturbations de petite amplitude, de sorte que la propagation des ondes puisse être considérée comme un processus isothermique.

3. En utilisant l'expression (2.6) de la vitesse Alfvén nous pouvons écrire l'équation de Hamilton-Jacobi sous la forme

$$(3.1) \quad \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = \frac{1}{U^2} = \frac{1}{U_0^2} (1 - 2\lambda z).$$

On remarque immédiatement que les variables x et y sont des variables cycliques; par conséquent nous pouvons mettre dans l'équation (3.1)

$$(3.2) \quad \frac{\partial S}{\partial x} = \alpha, \quad \frac{\partial S}{\partial y} = \beta.$$

Puisque les variables se séparent les équations (3.1) et (3.2) nous donnent

$$(3.3) \quad \begin{aligned} S &= \alpha x + \beta y + \int \sqrt{\frac{1}{U_0^2} (1 - 2\lambda z) - (\alpha^2 + \beta^2)} dz + C = \\ &= \alpha x + \beta y - \frac{U_0}{3\lambda} [2\lambda(z_0 - z)]^{3/2} + C, \end{aligned}$$

où par z_0 nous avons noté l'expression

$$(3.4) \quad z_0 = \frac{1 - U_0^2(\alpha^2 + \beta^2)}{2\lambda}.$$

On remarque que S est réel seulement pour des valeurs de z inférieurs à une valeur limite

$$(3.5) \quad z < z_0.$$

Selon (3.3) et (3.4), l'équation des rayons résultera du système

$$(3.6) \quad \begin{cases} \frac{\partial S}{\partial \alpha} = x + \frac{U_0}{\lambda} [2\lambda(z_0 - z)]^{1/2} \alpha = x_0, \\ \frac{\partial S}{\partial \beta} = y + \frac{U_0}{\lambda} [2\lambda(z_0 - z)]^{1/2} \beta = y_0. \end{cases}$$

Ainsi, après quelques transformations élémentaires, le système (3.5) devient

$$(3.7) \quad \begin{cases} \frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta}, \\ z_0 - z = \lambda r^2 = \lambda [(x - x_0)^2 + (y - y_0)^2]. \end{cases}$$

Par conséquent, les trajectoires des rayons sont des courbes planes, résultat absolument normal et conforme au caractère plan du champ magnétique.

Dans la fig. 2. nous avons représenté les trajectoires des „rayons“ ayant choisi comme origine du repère de coordonnées le point $(x_0, y_0, 0)$.

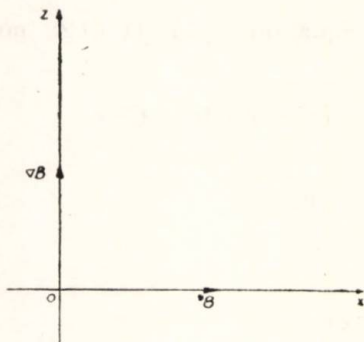


Fig. 1

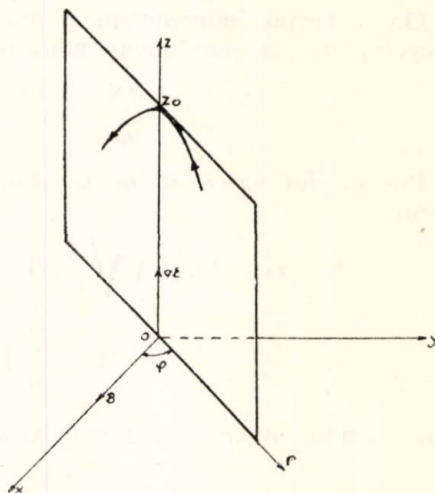


Fig. 2

Nous remarquons, d'abord, qu'il y a une famille de surface caractéristiques (en espèce les plans $z = \text{const}$) qui limite la pénétration des ondes magnétoacoustiques dans les domaines à champ magnétique croissant. Sur ces surfaces les „rayons“ sont réfléchis totalement, le vecteur d'onde $\vec{k}$ restant, naturellement, dans le plan d'incidence.

En outre, on remarque que l'enveloppe de la famille des surfaces à deux paramètres (x_0, y_0) donnée par la relation (3.7) est le plan $z = z_0$.

4. Supposons, dans le plan $z = 0$, une source d'ondes magnétoacoustiques qui émet uniformément dans toutes les directions. En l'occurrence, la pente de la trajectoire du „rayon“ dans le point où ce „rayon“ passe à travers le plan $z = 0$ sera

$$(4.1) \quad \operatorname{tg} \vartheta = \frac{dz}{dr} = -2\lambda r_0 = -\sqrt{4\lambda z_0},$$

d'où

$$(4.2) \quad z_0 = \frac{\operatorname{tg}^2 \vartheta}{4\lambda}.$$

Conformément à la dernière relation, on peut conclure que les ondes émises par la source sous certains angles sont réfléchis sur certains plans $z = z_0$. Par conséquent, le système décrit joue, dans un certain sens, le rôle d'un système dont la transparence pour les ondes magnétoacoustiques dépend de la direction dans laquelle ces ondes ont été émises. Évidemment

ce sont les ondes émises dans la direction normale au plan dans lequel se trouve la source, qui jouissent d'une transmission préférentielle.

Maintenant, nous allons modifier la géométrie du champ magnétique en supposant que celui-ci est symétrique par rapport au plan de la source ($z = 0$), ses valeurs dans le demiplan supérieur $z \geq 0$ restant les mêmes.

En l'occurrence, les résultats obtenues ci-dessus restent valables. En outre, on remarque que, de chaque côté du plan de la source il y aura une surface réfléchissante, ainsi que la propagation des ondes émises par la source soit limitée de deux côtés (fig. 3).

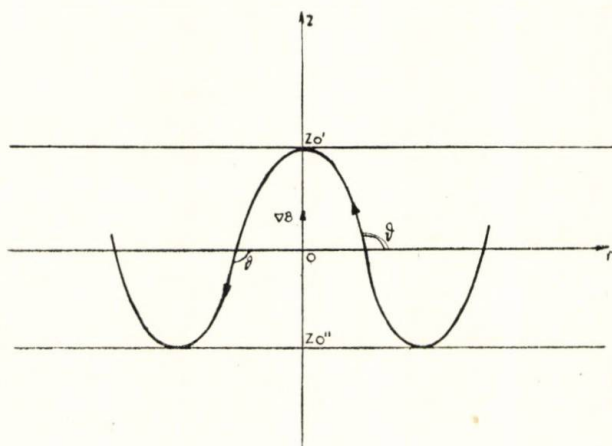


Fig. 3

Puisque les rayons émis sous un certain angle et réfléchis sur le plan $z = z_0'$ traverseront le plan de la source sous le même angle, les deux plans limite seront symétriques par rapport au plan $z = 0$ ($z_0' = z_0''$).

On remarque que le système tellement modifié se conduit, en certaine mesure, comme un guide d'ondes pour les perturbations magnétoacoustiques émises par la source dans divers directions.

Reçue le 18 janvier 1971

Institut Polytechnique, Jassy,
Chaire de Physique

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ASUPRA PROPAGĂRII UNDELOR MAGNETOACUSTICE ÎNTR-O MAGNETOPLASMĂ CU NEOMOGENITĂȚI LINIARE, AFLATĂ ÎN ECHILIBRU

(Rezumat)

Folosind metodele opticii geometrice se studiază propagarea undelor magnetoacustice într-un câmp magnetic cu neomogenități liniare. Se constată existența unor domenii interzise pentru aceste unde în regiunile cu câmp magnetic mai intens, adâncimea de pătrundere a lor depinzând de unghiul sub care au fost emise de către o sursă aflată într-un plan perpendicular pe ∇B . În cazul unei configurații cu un plan de simetrie, se arată că acesta joacă, într-o anumită măsură, rolul unui ghid de unde magnetoacustice.

STUDIUL STRUCTURII STRATURILOR SUBȚIRI DE Te PRIN MICROSCOPIE ELECTRONICĂ

DE

SOFIA MELINTE și M. LEANCĂ

1. Introducere

Studiul structurii straturilor subțiri prin microscopie electronică a făcut obiectul a numeroase lucrări, însă datorită multiplilor parametri care determină această structură, problema nu este elucidată. Datele cu privire la structura straturilor subțiri de Te lipsesc cu desăvârșire. Unii autori ca M. Gandais [1], [2] au studiat structura straturilor de Au, Ag, Cu, Bi, Antimoniu, arătând că aspectul lor diferă de la metal la metal și depinde de natura suportului, viteza de depunere, presiunea din incinta de vaporizare etc. Deoarece structura este cea care determină proprietățile optice, electrice, fotoelectrice ale straturilor, în această lucrare vom determina parametrii ce determină această structură. M. Perrot și colab. [3] și P. Larroque [7] indică o metodă directă de determinare a parametrilor de structură care influențează proprietățile optice ale straturilor metalice foarte subțiri, plecând de la numărarea directă a cristalitelor prin microscopie electronică. Este binecunoscut faptul că factorii de reflexie, transmisie și absorpție au fost determinați de diverși autori [4]...[6] în ipoteza că straturile sînt constituite din microcristalite sferice sau elipsoidale, care constituie un dipol electrizat pentru u. e. m. incidentă. Vom folosi aici metoda utilizată de [3] și [7], presupunînd că τ reprezintă suprafața acoperită de cristalite pe unitatea de suprafață a suportului, și vom admite că cristalitele au formă elipsoidală de revoluție cu axe a și b (a — axa de revoluție $\perp$ la suport și $b \parallel$ la suport) f fiind funcția de distribuție a raportului a/b ; $G(f)$ — funcția de distribuție a granulelor pe suport; $G(f)df$ — volumul metalului depus, aparținînd elipsoizilor de axe a și b , astfel că funcția f ia o valoare cuprinsă între f și $f + df$.

Raportînd la unitatea de suprafață a suportului găsim

$$(1) \quad \int G(f)df = e_m.$$

Deci modelul stratului granular studiat este determinat de grosimea masică e_m , de suprafața τ și de $G(f)$.

2. Metoda experimentală

Microscopia electronică necesită obiecte foarte mici, și deci stratul nu poate fi observat pe suportul lui de origine. Grosimea stratului nu trebuie să fie prea mare deoarece fascicolul de electroni îl distruge. Tehnica de lucru utilizată a fost următoarea: pe o lamelă de microscop se așează grila și se introduce într-un vas cu apă distilată; cu o baghetă se depun pe suprafața apei câteva picături de coloxilin. După evaporare rămîne pe suprafața apei o peliculă fină. Se scoate apa (cu mare atenție pentru a evita formarea curenților care ar face pliuri peliculei), pînă ce pelicula rămîne pe grilă. Apoi se usucă la o temperatură de cel mult 60°C (pentru a nu se deforma) și se depune stratul de metal. Vidul sub care s-a efectuat depunerea a fost de ordinul $10^{-5} \dots 10^{-6}$ torr. Pentru straturi groase s-a folosit tehnica replicilor; un strat foarte subțire de carbon s-a vaporizat în vid pe depozitul de pe sticlă și s-a luat amprenta. Detașarea replicii s-a făcut în acid sulfuric sau nitric; după imersarea în apă distilată replica s-a uscat și apoi s-a studiat la microscop.

3. Determinarea directă a parametrilor de structură

Se consideră o suprafață s din suportul studiat și se determină prin măsurare și numărare directă numărul n_i de cristalite ale căror diametre paralele la suport se notează cu b_i și lărgimea db_i . Valoarea medie a lor va fi

$$\bar{b} = \sum \frac{n_i b_i}{N},$$

unde

$$N = \sum n_i$$

și

$$\sigma = \Delta b = \sqrt{n_i(b_i - \bar{b})^2 / N}.$$

Atunci

$$(2) \quad n_i = \left(\frac{N}{\sqrt{2\pi}\sigma} \right) e^{-\frac{(b - \bar{b})^2}{2\sigma^2}} db.$$

Dacă se presupune că granulele sînt elipsoizi de revoluție în jurul axei normale la suport, se poate atunci calcula suprafața τ ocupată de metal pe unitatea de suprafață a suportului, prin relația:

$$(3) \quad \tau = \frac{1}{S} \frac{\pi}{4} \sum n_i b_i^2.$$

Această metodă nu dă distribuția diametrilor normali la suport (a). Cunoscîndu-se însă volumul metalului depus pe unitatea de suprafață, adică grosimea masică, se poate stabili relația dintre aceste două mărimi anume

$$e_m = \frac{1}{S} \frac{\pi}{6} \sum n_i b_i^2 \bar{a},$$

de unde

$$(4) \quad \bar{a} = 1,5 \frac{e_m}{\tau}.$$

Cunoscînd b și a se poate calcula valoarea funcției $f(b/a)$.

În cazul sferei $f = 0,33$ deoarece $a = b$. Pentru elipsoid aplatizat (cazul straturilor mai groase) $a < b, f < 0,33$; pentru elipsoid alungit $a > b, f > 0,33$.

Pentru alte forme ale cristalitelor, cilindru, sferă; care sînt mai simple,

$$(5) \quad \bar{a} = C \frac{e_m}{\tau},$$

unde C este un factor de formă egal cu 1 pentru cilindru și superior lui 1, pentru alte forme.

4. Rezultate experimentale

Pentru fig. 1 și 2, cristalitele fiind sferice ($a = b$), s-au determinat parametrii redați în tabelul 1.

Tabelul 1

Fig.	e_m $m\mu$	b $m\mu$	$\bar{b}$ $m\mu$	τ
1	0,78	2,36	1,81	0,045
2	2,05	3,75	2,68	0,164

Se constată că pe măsură ce grosimea crește, diametrul cristalitelor crește ocupînd o suprafață mai mare. În cazul din fig. 3, $d = 3,5 m\mu$, cristalitele au crescut în diametru și unindu-se au format punți. Fig. 4 reprezintă

un strat de grosime $d = 9,5 \text{ m}\mu$, care permite să se constate că în cazul stratului mai gros, cristalitele sînt aplatizate, $a < b$, cu numeroase centre de condensare (punctele mai întunecate).

Dacă s-ar reprezenta $n_i = f(b)$ am obține histogramele care ajustate au forma clopotului lui Gauss,

După cum rezultă din figuri, nu există o orientare a cristalitelor. Originea dezorientării cristalitelor pe suportul de coloxilin se datorește plasticității suportului, care antrenează foarte mari deformații ale suprafeței. La aceasta contribuie și forțele de adeziune ale atomilor, care determină pătrunderea atomilor între molecule. P. Benjamin [8], măsurînd energia de formare a straturilor subțiri, arată că forțele de aderență a atomilor la suport sînt forțe Van der Waals. S-a admis în general ideea că în cazul evaporării metalelor sub vid, aderența metal-sticlă se face prin intermediul unui oxid metallic, lucru ce pare verosimil.

Atomii absorbiți chimic constituie germeni pentru formarea unui cristal. Așa se explică apariția pe fig. 4 a unor aglomerate mai proeminente.

4.1. Difrakția electronilor

Studiul structurii straturilor efectuat prin difracție electronică arată că atunci cînd stratul este uniform inelele sînt cercuri perfecte și continui. Din analiza celor două fotografii efectuate pentru un strat cu grosimea masică $e_m = 3,6 \text{ m}\mu$ și $e_m = 9,5 \text{ m}\mu$ rezultă că în fig. 5 difracția electronilor s-a produs prin transmisie. Forma inelelor de difracție este foarte mult influențată de structura stratului. Inelele apar întrerupte, în număr mare, deoarece stratul este subțire, deci suficient de transparent. Lățimea inelelor se datorește refracției electronilor pe diferite cristalite. Faptul că diagrama este formată din inele perfecte denotă că cristalitele sînt orientate și stratul este uniform.

Fig. 6 reprezintă diagrama de difracție a stratului de grosime masică $e_m = 9,5 \text{ m}\mu$. Se observă că inelele sînt continui, mult mai dese și vizibil în număr mai mic. Acest lucru dovedește că pe măsură ce crește în grosime, stratul este mult mai uniform și că aranjarea defectuoasă apare în general astfel: la bază cristalitele se orientează după axa 111 normală la suport, deasupra se produce aranjarea defectuoasă, ca apoi să reapară prima orientare.

5. Concluzii

1. Studiul efectuat prin microscopie electronică confirmă presupunerile făcute de mulți autori, printre care E. David, G. Rassigni ș. a., că straturile sînt formate din cristalite și că aceste cristalite au forma sferică sau elipsoidală.

2. Structura uniformă sau neuniformă a stratului este determinată de foarte mulți parametri, dintre care esențiali sînt: presiunea din in-

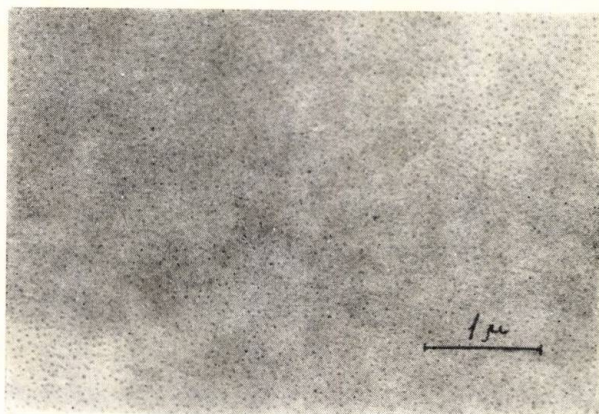


Fig. 1

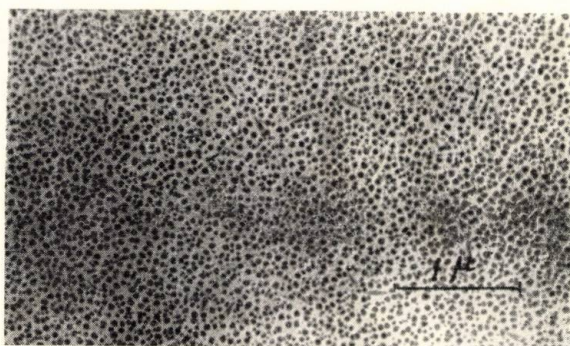


Fig. 2

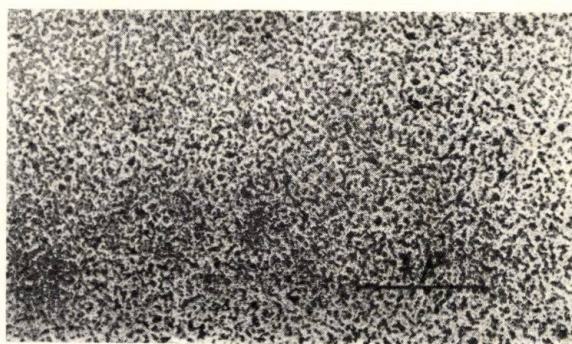


Fig. 3

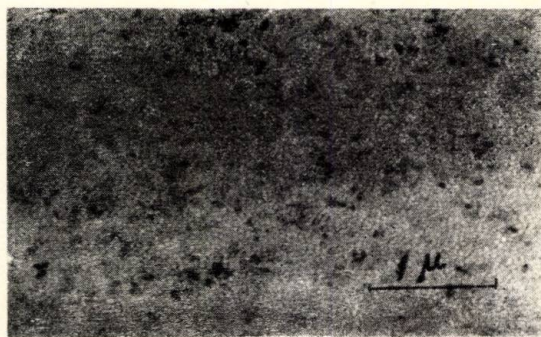


Fig. 4



Fig. 5

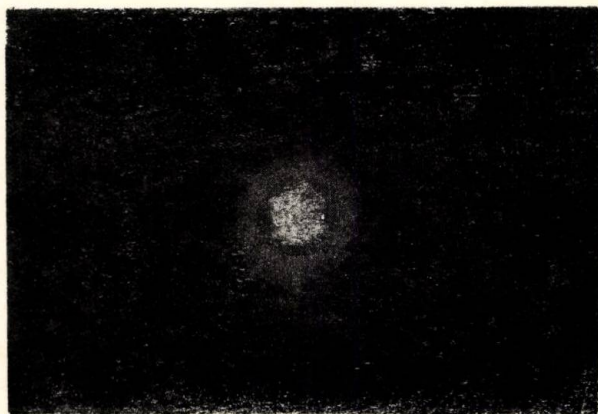


Fig. 6

cinta de vaporizare, natura și starea de curățenie a suportului, viteza de vaporizare etc.

Această structură influențează în mod esențial proprietățile optice ale straturilor subțiri.

Primită la 5 noiembrie 1970

Institutul politehnic, Iași,
Catedra de fizică

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L'ÉTUDE DE LA STRUCTURE DES COUCHES MINCES DE Te À L'AIDE DE LA MICROSCOPIE ÉLECTRONIQUE

(Résumé)

On étudie la structure des couches minces de Te par microscopie et diffraction électronique.

On détermine les paramètres structuraux des couches minces, en présentant aussi une méthode directe et une méthode de préparation des éprouvettes qui permettent leur étude au microscope électronique.

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